TENSOR PRODUCTS, CHARACTERS, AND BLOCKS OF FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS AT ROOTS OF UNITY

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Abstract: We establish several results concerning tensor products, ℓ-characters, and the block decomposition of the category of finite-dimensional representations of a quantum affine algebra in the root of unity setting. We also give a counterexample showing that Weyl modules may not be isomorphic to a tensor product of fundamental representations in this setting. In the generic case, this isomorphism was essential for establishing the block decomposition theorem. We overcome the lack of such tool in the root of unity setting by using results on specialization of modules.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over the complex numbers and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ its associated loop algebra. Following Drinfeld and Jimbo, one can consider the quantum groups $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ which are Hopf algebras over the field $\mathbb{C}(q)$ of rational functions in the indeterminate q. The latter is most often called a quantum affine algebra. Using $\mathbb{C}[q, q^{-1}]$-forms of $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$, one defines specializations of the quantum groups at $q = \xi \in \mathbb{C}\backslash\{0\}$. If ξ is generic, i.e., not a root of unity, the "general behavior" of the resulting algebras does not depend on ξ or on the chosen form. However, if ξ is a root of unity, the resulting algebras depend drastically on the chosen form, as well as on ξ.

In this paper, we focus on the algebras obtained from Lusztig's form which is the one generated by the divided powers of the quantum Chevalley generators of $U_q(\mathfrak{g})$ or $U_q(\tilde{\mathfrak{g}})$. The resulting algebras will be denoted by $U_\xi(\mathfrak{g})$ and $U_\xi(\tilde{\mathfrak{g}})$, respectively. We study the category $\widetilde{\mathcal{C}}_\xi$ of (type 1) finite-dimensional representations of $U_\xi(\tilde{\mathfrak{g}})$ with special attention to the root of unity setting (the order of the root of unity is assumed to be odd and relatively prime to the lacing number of $\mathfrak{g}$). We start by studying the concept of Weyl modules which had not been considered in the literature before in the root of unity context. Next, we obtain several results concerning specialization of modules and prove a version of the main result of [8] which gives a sufficient condition for a tensor product of simple objects of $\widetilde{\mathcal{C}}_\xi$ to be a quotient of the appropriate Weyl module. In the generic case, this result combined with results of Beck and Nakajima [5] were used in [12] to prove that the Weyl modules are isomorphic to certain tensor products of fundamental modules. We give an example showing that this is not always the case in the root of unity setting. We proceed by describing the block decomposition of $\widetilde{\mathcal{C}}_\xi$. Finally, we show that the main result of [12] regarding the braid group invariance of the ℓ-characters of fundamental representations holds in the root of unity context as well.

The block decomposition of the category $\widetilde{\mathcal{C}}_\xi$ was first studied in [22] where it was proved that the blocks are parameterized by the so-called elliptic characters which are elements of the quotient of the ℓ-weight lattice of $U_\xi(\tilde{\mathfrak{g}})$ by the ℓ-root lattice. However, in that paper ξ was assumed to be a nonzero complex number satisfying $|\xi| \neq 1$. The reason behind this assumption was that the original approach of [22] used analytic properties of the action of the R-matrix of $U_\xi(\tilde{\mathfrak{g}})$ which could be guaranteed only for such ξ. A new approach, in connection with the ℓ-characters (more often known as q-characters),

was developed in [12]. This approach was used to describe the blocks of $\widetilde{\mathcal{C}}_\xi$ for all generic ξ. We will mostly follow this later approach to describe the blocks of $\widetilde{\mathcal{C}}_\xi$ in the case that ξ is a root of unity. However, our results concerning tensor products show that one of the main techniques used in the generic context is not available in the root of unity one. Namely, an essential tool for proving that two objects having the same elliptic character must be in the same block was a corollary of [8, Theorem 4.4] stating that a tensor product of fundamental representations can always be reordered in such a way that it is a quotient of the corresponding Weyl module. In fact, as mentioned above, it was shown in [12] that such reordered tensor products are isomorphic to the corresponding Weyl module thus proving that every Weyl module is isomorphic to a tensor product of fundamental representations. The latter statement, as well as the aforemention corollary, is false (see Example 4.25) in the root of unity setting. Because of this, our version of [8, Theorem 4.4] in the root of unity context (Theorem 4.33) requires a somewhat restrictive hypothesis. In any case, even if such hypothesis could be removed (which is the case for $\mathfrak{g}$ of type A_1), the technique of tensor products could not be used as in the generic context.

To overcome this issue, we consider specialization of modules. Namely, we prove that the irreducible quotient of the specialization of an irreducible constituent of a Weyl module is an irreducible constituent of the specialization of that Weyl module. Using this and combinatorial arguments, we are then able to "lift" simple objects having the same elliptic character from the root of unity to the formal q context in such a way that the modules thus obtained have the same elliptic character again. We then use the results of [12] showing that these lifted modules are linked by a certain sequence of Weyl modules. By specializing this sequence back to the root of unity context we are then able to show that the starting two modules must have been in the same block. We remark that this proof also works for $\xi = 1$ in which case $\widetilde{\mathcal{C}}_\xi$ is the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules. The blocks for $\xi = 1$ were described in [11] using a different approach (for showing that two objects with the same elliptic character must be in the same block). Hence, our proof is an alternate one to the proof of the block decomposition theorem in the classical context as well. In fact, the proof works for any generic value of ξ and, therefore, it can be thought of as a uniform proof. We note that the elliptic characters were called spectral characters in [11] since their elliptic behavior degenerated when $\xi = 1$.

The theory of ℓ-characters is one of the most interesting topics of the finite-dimensional representation theory of quantum affine algebras. It was initiated in [26] under the name q-characters. There are several papers dedicated to the study of ℓ-characters of the finite-dimensional irreducible $U_\xi(\tilde{\mathfrak{g}})$-modules and other important subclasses of representations such as standard or Weyl modules (see [24, 25, 27, 28, 42, 43, 44] and references therein). The ℓ-characters can be encoded in a ring homomorphism from the Grothendieck ring R_ξ of $\widetilde{\mathcal{C}}_\xi$ to the integral group ring $\mathbb{Z}[\mathcal{P}_\xi]$ of the ℓ-weight lattice. In [42], under the assumption that $\mathfrak{g}$ is simply laced, Nakajima defined certain polynomials similar to Kazhdan-Lusztig polynomials by studying cohomology of quiver varieties. This lead him to define a function $\chi_{\xi,t} : R_\xi \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[\mathcal{P}_\xi] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$ called the t-analogue of the ℓ-character ring homomorphism. It turns out that at $t = 1$ this function specializes to the ℓ-character ring homomorphism. Moreover, the definition of $\chi_{\xi,t}$ was axiomatized in a purely combinatorial manner leading to an algorithm for computing the ℓ-characters of the irreducible or of the standard modules. This algorithm was used in [43] to give explicit formulas for the t-analogues of the ℓ-characters of the standard modules when $\mathfrak{g}$ is of type A or D and ξ is not a root of unity. The formulas were presented in terms of tableaux and a connection with the theory of crystals was discovered. The algorithm was also used with the help of a supercomputer to compute the t-analogues of the ℓ-characters of the fundamental modules when $\mathfrak{g}$ is of type E in [44] (also for generic ξ). In [27, 28], Hernandez proved a conjecture of Nakajima saying that the existence of the function $\chi_{\xi,t}$ could be established using only its axiomatic description (without the use of geometry). This allowed him to extend the concept of t-analogues of ℓ-characters for general $\mathfrak{g}$. However, due to the lack of a definition of the quiver varieties in general, a

proof that the algorithm indeed gives the ℓ-characters of the irreducible modules when $\mathfrak{g}$ is not simply laced $\mathfrak{g}$ is still missing.

In [12, Theorem 6.1], Chari and the second author proved that the ℓ-characters of the fundamental representations satisfy a certain invariance property with respect to the braid group action on the ℓ-weight lattice provided $\mathfrak{g}$ is of classical type and ξ is not a root of unity. As mentioned above, we show that this remains valid in the root of unity setting as well (see Theorem 4.47). This result was used in [13] to obtain closed formulas for the ℓ-characters of fundamental representations of $U_\xi(\tilde{\mathfrak{g}})$. Actually, this theorem gives a recursive method for computing a lower bound for the ℓ-characters of the fundamental representations. In order to show that this is also an upper bound, the authors of [13] used that the character of the fundamental representations (i.e., their $U_\xi(\mathfrak{g})$-structures) were already known. By looking at the simplest nontrivial example, we show that this procedure may be applied in the root of unity setting as well. Namely, we use Theorem 4.47 and the theory of specialization of modules to obtain an explicit formula for the ℓ-character of the fundamental representation corresponding to the adjoint representation when $\mathfrak{g}$ is of type D_n and ξ is any root of unity (of odd order) in terms of the braid group action on the ℓ-weight lattice. In particular, we show that this fundamental module is irreducible as $U_\xi(\mathfrak{g})$-module iff the order of ξ divides $n-2$ (Nakajima has explained to us that the same result can be deduced from the algorithm of [42]). We believe that the same line of reasoning can be used in general but, since the combinatorics would be rather lengthy, we find it more appropriate to leave this task for a forthcoming publication. We remark, however, that there are more ℓ-weight spaces with multiplicity higher than one in the root of unity setting than in the generic one as it can be seen in the example we considered here. We explicitly describe the multiplicities in this example (see (4.29)).

The paper is organized as follows. In Section 1, we fix the basic notation and review the construction and some structural results for the algebra $U_\xi(\tilde{\mathfrak{g}})$. In Section 2, we review the definitions of the ℓ-weight and the ℓ-root lattices and describe their quotient. Also in Section 2, we give the definitions of resonant order for dominant ℓ-weights and of ξ-regular dominant ℓ-weights which are the main combinatorial conditions used in the statement of Theorem 4.33. Section 3 brings the basic facts of the finite-dimensional representation theory of $U_\xi(\tilde{\mathfrak{g}})$ such as the classification of the irreducible modules in terms of dominant ℓ-weights, the existence of universal finite-dimensional highest-ℓ-weight modules called the Weyl modules, and a few results related to the ℓ-characters. We remark that the proof of the existence of the Weyl modules was not complete in the literature for type G_2. We provide the missing technical details for this case here. The main results of the paper are in Section 4. In §4.1, §4.2, and §4.3 we review the basics of the theory of specialization of modules, evaluation representations, and the existence of the Frobenius homomorphism. We remark, however, that to the best of our knowledge the main result of §4.1 (Theorem 4.3) had not been considered in the literature if $\mathfrak{g}$ is of type G_2. It turns out that this case has some extra technical difficulties which we have resolved here. In §4.4 we prove the aforementioned Theorem 4.33 as well as a stronger version of it for two-fold tensor products in the case $\mathfrak{g} = \mathfrak{sl}_2$. In §4.5 we prove that, under mild assumptions on the highest ℓ-weight, the specialization of an irreducible constituent of a highest-ℓ-weight module is an irreducible constituent of the specialization of that module. We use this to prove Theorem 4.38 which says that every simple object of $\widetilde{\mathcal{C}}_\xi$ is a constituent of the specialization of a highest-ℓ-weight tensor product of fundamental representations with ℓ-fundamental weights indexed by $I_\bullet$ (see Table 1 below). We remark that if ξ is not a root of unity one can remove the expression "the specialization of" from the statement (this is a consequence of Theorem 4.22 among other things), but in the root of unity setting that is not the case as shown by Example 4.25. The description of the blocks is given in §4.6 and the results concerning the ℓ-characters of fundamental representations is given in §4.7.

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1. The Quantum Loop Algebras at Roots of Unity

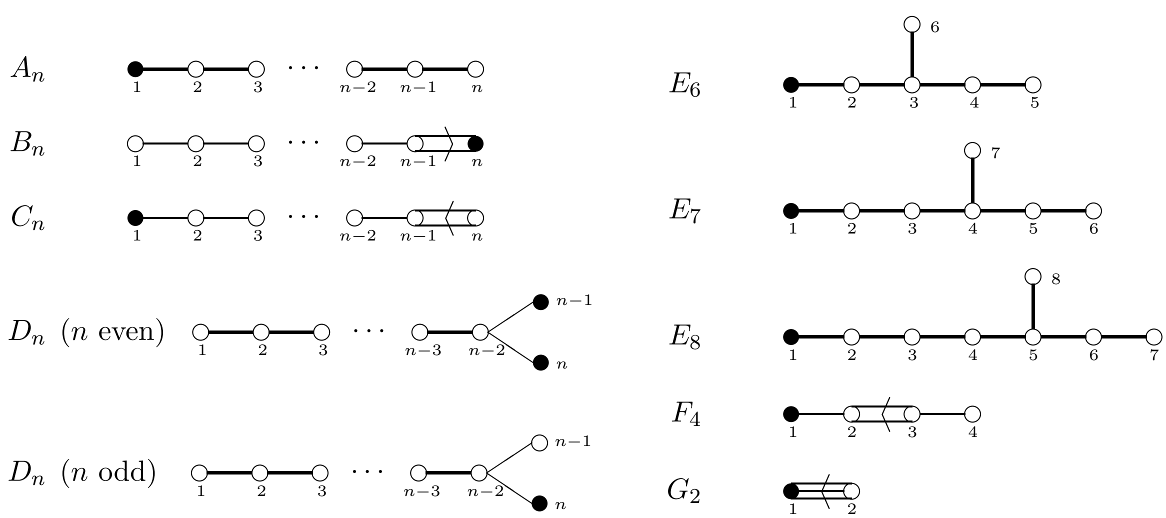
1.1. **Notation and Basics on Simple Lie Algebras.** Throughout, $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_{\geq 0}$ denote the sets of complex numbers, integers, and non-negative integers. For any integer m, $\mathbb{Z}_{>m}$ denotes the set of all integers greater than m. Given a ring $\mathbb{A}$, the underlying multiplicative group of units is denoted by $\mathbb{A}^\times$. The dual of a vector space V is V^*. The symbol "$\cong$" denotes isomorphisms. Tensor products of vector spaces are always taken over the underlying field unless otherwise indicated.

Let $\mathfrak{g}$ be a simple Lie algebra of rank n over the complex numbers with a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

Let $I = \{1, \ldots, n\}$ be an indexing set of the vertices of the Dynkin diagram of $\mathfrak{g}$ and R^+ the set of positive roots. The simple roots are denoted by α_i, the fundamental weights by ω_i, while Q, P, Q^+, P^+ denote the root and weight lattices with corresponding positive cones, respectively. Let also h_i be the coroot associated to α_i and denote by $x_i^\pm$ a basis element of the root space corresponding to $\pm\alpha_i, i \in I$. Equip $\mathfrak{h}^*$ with the partial order $\lambda \leq \mu$ iff $\mu - \lambda \in Q^+$. Denote by $\mathcal{W}$ the Weyl group of $\mathfrak{g}$, by s_i the simple reflections, by $\ell(w)$ the length of $w \in \mathcal{W}$, and let w_0 be the longest element of $\mathcal{W}$. Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, i.e., $c_{ij} = \alpha_j(h_i)$, and let $D = \mathrm{diag}(d_i : i \in I)$ where the numbers d_i are coprime positive integers such that DC is symmetric. We suppose the nodes of the Dynkin diagram of $\mathfrak{g}$ are labeled as in Table 1 below and let $I_\bullet$ be the indexing set of the black nodes.

Table 1



We shall need the following well-known lemma which can be deduced from the results of [29].

Lemma 1.1. Let $\lambda \in P^+$.

(a) If $\mu \in P^+$ is such that $\mu \leq \lambda$ and $w \in \mathcal{W}$, then $w\mu \leq \lambda$. Moreover, $w_0\lambda$ is the unique minimal element of the set $P(\lambda) := \{w\mu : w \in \mathcal{W}, \mu \in P^+, \mu \leq \lambda\}$.

(b) If $i \in I$ and $w \in \mathcal{W}$ are such that $\ell(s_i w) = \ell(w) + 1$, then $w^{-1}\alpha_i \in R^+$. In particular, $w\lambda + \alpha_i \notin P(\lambda)$. $\square$

If $\mathfrak{a}$ is a Lie algebra over $\mathbb{C}$, define its loop algebra to be $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ with bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$. Clearly $\mathfrak{a} \otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to $\mathfrak{a}$ and, by abuse of notation, we will continue denoting its elements by x instead of $x \otimes 1$. We have $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra. The elements $x_i^\pm \otimes t^r$ and $h_i \otimes t^r$ will be denoted by $x_{i,r}^\pm$ and $h_{i,r}$, respectively.

Let $U(\mathfrak{a})$ denote the universal enveloping algebra of a Lie algebra $\mathfrak{a}$. Then $U(\mathfrak{a})$ is a subalgebra of $U(\tilde{\mathfrak{a}})$. Moreover, if $\mathfrak{a}$ is a direct sum of two of its subalgebras, say $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c}$, then multiplication establishes an isomorphism of vector spaces $U(\mathfrak{b}) \otimes U(\mathfrak{c}) \to U(\mathfrak{a})$. The assignments $\triangle : \mathfrak{a} \to U(\mathfrak{a}) \otimes U(\mathfrak{a}), x \mapsto x \otimes 1 + 1 \otimes x$, $S : \mathfrak{a} \to \mathfrak{a}, x \mapsto -x$, and $\epsilon : \mathfrak{a} \to \mathbb{C}, x \mapsto 0$, can be uniquely extended so that $U(\mathfrak{a})$ becomes a Hopf algebra with comultiplication $\triangle$, antipode S, and counit ϵ. Given a Hopf algebra H, we shall denote by H^0 the augmentation ideal of H, i.e., the kernel of its counit.

For each $i \in I$ and $r \in \mathbb{Z}$, define elements $\Lambda_{i,r} \in U(\tilde{\mathfrak{h}})$ by the following equality of formal power series in the variable u:

$$\Lambda_i^\pm(u) = \sum_{r=0}^\infty \Lambda_{i,\pm r} u^r = \exp\left(- \sum_{s=1}^\infty \frac{h_{i,\pm s}}{s} u^s \right). \tag{1.1}$$

1.2. **Quantum Loop Algebras.** Let $\mathbb{C}(q)$ be the ring of rational functions in an indeterminate q and define

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_q! = [m]_q[m-1]_q \ldots [2]_q[1]_q, \qquad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q \cdots [m-r+1]_q}{[r]_q!},$$

for $r, m \in \mathbb{Z}_{\geq 0}$. If $m \geq r$, $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_q = \dfrac{[m]_q!}{[r]_q![m-r]_q!}$.

Set $\mathbb{A} = \mathbb{C}[q, q^{-1}]$ and recall that $[m]_q, [m]_q!, \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_q \in \mathbb{A}$. Thus, when q specializes to a non-zero complex number ζ, then $[m]_q, [m]_q!, \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_q$ specialize to complex numbers which will be denoted by $[m]_\zeta, [m]_\zeta!, \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_\zeta$, respectively.

Set $q_i = q^{d_i}$ and $[m]_i = [m]_{q_i}$. The quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ of $\mathfrak{g}$ is the algebra over $\mathbb{C}(q)$ with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z} \backslash \{0\}$) and the following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i h_{j,r} = h_{j,r} k_i, \quad h_{i,r} h_{j,s} = h_{j,s} h_{i,r},$$
$$k_i x_{j,r}^\pm k_i^{-1} = q_i^{\pm c_{ij}} x_{j,r}^\pm, \qquad [h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r c_{ij}]_i x_{j,r+s}^\pm,$$
$$x_{i,r}^\pm x_{j,s}^\pm - q_i^{\pm c_{ij}} x_{j,s}^\pm x_{i,r}^\pm = q_i^{\pm c_{ij}} x_{i,r-1}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r-1}^\pm,$$
$$[x_{i,r}^+, x_{j,s}^-] = \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q_i - q_i^{-1}},$$
$$\sum_{\sigma \in S_m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_i x_{i,r_{\sigma(1)}}^\pm \ldots x_{i,r_{\sigma(k)}}^\pm x_{j,s}^\pm x_{i,r_{\sigma(k+1)}}^\pm \ldots x_{i,r_{\sigma(m)}}^\pm = 0, \quad \text{if } i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - c_{ij}$, S_m is the symmetric group on m letters, and $\psi_{i,r}^\pm$ are determined by equating powers of u in the following equality of formal power series:

$$\Psi_i^\pm(u) := \sum_{r=0}^\infty \psi_{i,\pm r}^\pm u^r = k_i^{\pm 1} \exp\left(\pm (q_i - q_i^{-1}) \sum_{s=1}^\infty h_{i,\pm s} u^s \right).$$

Denote by $U_q(\tilde{\mathfrak{n}}^\pm)$, $U_q(\tilde{\mathfrak{h}})$ the subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by $\{x_{i,r}^\pm\}, \{k_i^{\pm 1}, h_{i,s}\}$, respectively. Let $U_q(\mathfrak{g})$ be the subalgebra generated by $x_{i,0}^\pm, k_i^{\pm 1}$ and define $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$ in the obvious way. Note that $\psi_{i,0}^\pm = k_i^{\pm 1}$ and that $U_q(\mathfrak{g})$ is the quantum group as defined in [38]. Henceforth, we may denote $x_{i,0}^\pm$ by $x_i^\pm$. Multiplication establishes isomorphisms of $\mathbb{C}(q)$-vectors spaces:

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \qquad \text{and} \qquad U_q(\tilde{\mathfrak{g}}) \cong U_q(\tilde{\mathfrak{n}}^-) \otimes U_q(\tilde{\mathfrak{h}}) \otimes U_q(\tilde{\mathfrak{n}}^+) .$$

We will also consider the subalgebra $U_q(\tilde{\mathfrak{g}}_i)$ generated by $k_i^{\pm 1}, h_{i,r}, x_{i,s}^\pm$ for all $r, s \in \mathbb{Z}, r \neq 0$. The algebra $U_q(\tilde{\mathfrak{g}}_i)$ is isomorphic to $U_{q_i}(\tilde{\mathfrak{sl}}_2)$. The subalgebra $U_q(\mathfrak{g}_i)$ is defined similarly.

We shall make use of the following proposition whose proof is straightforward.

Proposition 1.2. Let $a \in \mathbb{C}(q)^\times$. Then, there exists a unique $\mathbb{C}(q)$-algebra automorphism ϱ_a of $U_q(\tilde{\mathfrak{g}})$ such that ϱ_a is the identity on $U_q(\mathfrak{g})$ and $\varrho_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm$. □

1.3. **Restricted Integral Form and Specialization.** We now recall the definition of Lusztig's restricted integral form $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ of $U_q(\tilde{\mathfrak{g}})$ and the corresponding specialization $U_\xi(\tilde{\mathfrak{g}})$ where ξ is a nonzero complex number (cf. [19, 25, 38]).

For $r, c \in \mathbb{Z}, s, m \in \mathbb{Z}_{>0}, i \in I$, let $(x_{i,r}^\pm)^{(m)} = \frac{(x_{i,r}^\pm)^m}{[m]_i!} \in U_q(\tilde{\mathfrak{g}})$ and

$$\begin{bmatrix} k_i; c \\ s \end{bmatrix} = \prod_{m=1}^{s} \frac{k_i q_i^{c+1-m} - k_i^{-1} q_i^{-c-1+m}}{q_i^m - q_i^{-m}}. \tag{1.2}$$

Denote $\left[\begin{smallmatrix} k_i;0 \\ s \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} k_i \\ s \end{smallmatrix}\right]$. Define elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$, of $U_q(\tilde{\mathfrak{g}})$ by

$$\Lambda_i^\pm(u) = \sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(-\sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]_i} u^s\right). \tag{1.3}$$

Note that

$$\Psi_i^\pm(u) = k_i^{\pm 1} \frac{\Lambda_i^\pm(u q_i^{-1})}{\Lambda_i^\pm(u q_i)} \tag{1.4}$$

where the division is that of formal power series in u with coefficients in $U_q(\tilde{\mathfrak{g}})$. Although we are denoting the elements $x_{i,r}^\pm, h_{i,r}, \Lambda_{i,r}$ of $U_q(\tilde{\mathfrak{g}})$ by the same symbols as their classical counterparts in $\tilde{\mathfrak{g}}$, this will not create confusion as it will be clear from the context. Notice also that $U_q(\tilde{\mathfrak{h}})$ is generated by $k_i^{\pm 1}$ and $\Lambda_{i,r}$, $i \in I, r \in \mathbb{Z}$, as a $\mathbb{C}(q)$-algebra.

Let $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ be the $\mathbb{A}$-subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the elements $(x_{i,r}^\pm)^{(m)}$ and k_i for all $i \in I, r \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$.

Theorem 1.3. We have $U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{A}} \mathbb{C}(q) \cong U_q(\tilde{\mathfrak{g}})$.

Proof. Let θ be the maximal root of $\mathfrak{g}$ and write $\theta = \sum_{i \in I} \theta_i \alpha_i$. If $\mathfrak{g}$ is such that $\theta_i = 1$ for some $i \in I$, the proof is essentially given in [4] (the proof is written down completely only in the case $\mathfrak{g}$ is simply laced but, as pointed out in the proof of [7, Lemma 2.1], the same proof goes through for $\mathfrak{g}$ of type B or C). In the three remaining cases (E_8, F_4, and G_2) we have $\min\{\theta_i : i \in I\} = 2$. Following the reasoning of the proof of [4, Proposition 2.6], part of the argument can be modified to cover these cases by using [37, Lemma 2.7]. □

Let $U_{\mathbb{A}}(\tilde{\mathfrak{n}}^\pm) = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\tilde{\mathfrak{n}})^\pm$ and $U_{\mathbb{A}}(\tilde{\mathfrak{h}}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\tilde{\mathfrak{h}})$. It was proved in [19, Proposition 6.1] that multiplication establishes an isomorphism of $\mathbb{A}$-modules

$$U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cong U_{\mathbb{A}}(\tilde{\mathfrak{n}}^-) \otimes U_{\mathbb{A}}(\tilde{\mathfrak{h}}) \otimes U_{\mathbb{A}}(\tilde{\mathfrak{n}}^+). \tag{1.5}$$

Moreover, $U_{\mathbb{A}}(\tilde{\mathfrak{n}}^{\pm})$ is the $\mathbb{A}$-subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $\{(x_{i,r}^{\pm})^{(m)} : i \in I, r \in \mathbb{Z}\}$ and $U_{\mathbb{A}}(\tilde{\mathfrak{h}})$ is the $\mathbb{A}$-subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $\{k_i, [{k_i \atop s}], \Lambda_{i,r} : i \in I, r \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}\}$. The subalgebras $U_{\mathbb{A}}(\mathfrak{g})$, $U_{\mathbb{A}}(\mathfrak{n}^{\pm})$, and $U_{\mathbb{A}}(\mathfrak{h})$ are defined similarly and the corresponding statements to the ones above are obtained analogously. We note that $[{k_i;c \atop m}] \in U_{\mathbb{A}}(\mathfrak{g})$ and record the following relations (cf. [36]):

$$(1.6)\qquad (x_i^+)^{(p)}(x_i^-)^{(r)} = \sum_{m=0}^{\min\{p,r\}} (x_i^-)^{(r-m)} \begin{bmatrix} k_i; 2m-p-r \\ m \end{bmatrix} (x_i^+)^{(p-m)}, \qquad p, r \in \mathbb{Z}_{\geq 0}$$

$$(1.7)\qquad \begin{bmatrix} k_i; c \\ m \end{bmatrix} (x_{i,r}^{\pm})^{(p)} = (x_{i,r}^{\pm})^{(p)} \begin{bmatrix} k_i; c \pm pc_{ij} \\ m \end{bmatrix}, p \in \mathbb{Z}_{\geq 0}$$

Given $\xi \in \mathbb{C}^{\times}$, denote by ϵ_ξ the evaluation map $\mathbb{A} \to \mathbb{C}$ sending q to ξ and by $\mathbb{C}_\xi$ the $\mathbb{A}$-module obtained by pulling-back ϵ_ξ. Set

$$(1.8)\qquad U_\xi(\mathfrak{a}) = U_{\mathbb{A}}(\mathfrak{a}) \otimes_{\mathbb{A}} \mathbb{C}_\xi,$$

for $\mathfrak{a} = \mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{n}}^{\pm}, \tilde{\mathfrak{h}}$. The algebra $U_\xi(\mathfrak{a})$ is called the restricted specialization of $U_q(\mathfrak{a})$ at $q = \xi$. We shall denote an element of the form $x \otimes 1 \in U_\xi(\mathfrak{a})$ with $x \in U_{\mathbb{A}}(\mathfrak{a})$ simply by x. If ξ is not a root of unity, the algebra $U_\xi(\tilde{\mathfrak{g}})$ is isomorphic to the algebra given by generators and relations analogous to those of $U_q(\tilde{\mathfrak{g}})$ with ξ in place of q and its representation theory is parallel to that of $U_q(\tilde{\mathfrak{g}})$. We shall be particularly interested in the cases $\xi = \zeta$ where ζ is a nontrivial root of unity and $\xi = 1$. In the former case, the algebras $U_\zeta(\mathfrak{a})$ are also known as Lusztig's quantum groups. As for the later:

Proposition 1.4 ([15, 40])**.** $U(\tilde{\mathfrak{g}})$ (resp. $U(\mathfrak{g})$) is isomorphic to the quotient of $U_1(\tilde{\mathfrak{g}})$ (resp. $U_1(\mathfrak{g})$) by the ideal generated by $k_i - 1$. $\square$

Corollary 1.5. The category of $U_1(\tilde{\mathfrak{g}})$-modules (resp $U_1(\mathfrak{g})$-modules) on which k_i acts as the identity operator for all $i \in I$ is equivalent to the category of all $\tilde{\mathfrak{g}}$-modules (resp. $\mathfrak{g}$-modules). $\square$

Henceforth, assume l is an odd integer ≥ 3 and assume l is relatively prime to 3 if $\mathfrak{g}$ is of type G_2. Let $\zeta \in \mathbb{C}$ be a primitive l-th root of unity and set $\zeta_i = \zeta^{d_i}$ for $i \in I$. The hypotheses on l imply that ζ_i^2 is a root of 1 of order l as well. In particular, l is minimal such that $[l]_{\zeta_i} = 0$. If $\mathfrak{g}$ is not of type G_2, let $\mathbb{C}'$ be the set consisting of q and all nonzero complex numbers which are not roots of unity of even order. If $\mathfrak{g}$ is of type G_2 we also exclude from $\mathbb{C}'$ the roots of unity whose orders are multiples of 3. Henceforth, for simplicity of notation, the expression "root of unity" will pertain only to nontrivial roots of unity and not to 1. Unless stated otherwise, ξ will always denote an arbitrary element of $\mathbb{C}'$ and we set $l = 1$ when ξ is not a root of unity for notational convenience. An element of $\mathbb{C}'$ which is not a root of unity will be referred to as a generic element (so 1 and q are generic). If $\xi \in \mathbb{C}'$ satisfies $\xi^k = 1$ for some $k \in \mathbb{Z}_{>0}$, then ξ is said to have finite order. Otherwise, ξ has infinite order. Thus, 1 is the only generic element of finite order.

Remark. Although several of the results that follow can be proved for even roots of unity as well, some of the proofs (and even the precise statements) require many more technicalities. For simplicity and uniformity, we shall keep the above made assumptions on ξ.

It is well known that $U_\xi(\tilde{\mathfrak{g}})$ is a Hopf algebra. However, if $\xi \neq 1$, an expression for the comultiplication Δ of $U_\xi(\tilde{\mathfrak{g}})$ in terms of the generators $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}$ is not known (the definition is given in terms of the Chevalley generators which we do not need to consider here). The following partial information obtained in [3, 4, 17, 21] suffices for our purposes (see also Lemma 7.5 of [19]). First we recall that

$$(1.9)\qquad \Delta(k_i) = k_i \otimes k_i \quad \text{for all } i \in I.$$

Let $X^{\pm}$ be the $\mathbb{C}(\xi)$-span of the elements $x_{i,r}^{\pm}$ for all $i \in I, r \in \mathbb{Z}$. Similarly, given $i \in I$, let $X^{\pm}(i)$ be the $\mathbb{C}(\xi)$-span of the elements $x_{j,r}^{\pm}$ for all $j \in I \backslash \{i\}, r \in \mathbb{Z}$.

Proposition 1.6. The restriction of Δ to $U_\xi(\tilde{\mathfrak{g}}_i)$ satisfies:

(a) $\Delta(\left[\begin{smallmatrix} k_i \\ r \end{smallmatrix}\right]) = \sum_{j=0}^{r} \left[\begin{smallmatrix} k_i \\ r-j \end{smallmatrix}\right] k_i^{-j} \otimes \left[\begin{smallmatrix} k_i \\ j \end{smallmatrix}\right] k_i^{r-j}$, for every $r \in \mathbb{Z}_{\geq 0}$.

(b) Modulo $U_\xi(\tilde{\mathfrak{g}})X^- \otimes U_\xi(\tilde{\mathfrak{g}})(X^+)^{(2)} + U_\xi(\tilde{\mathfrak{g}})X^- \otimes U_\xi(\tilde{\mathfrak{g}})X^+(i)$, we have

$$\Delta(x_{i,r}^+) = x_{i,r}^+ \otimes 1 + k_i \otimes x_{i,r}^+ + \sum_{j=1}^{r} \psi_{i,j}^+ \otimes x_{i,r-j}^+ \quad (r \geq 0),$$

$$\Delta(x_{i,-r}^+) = k_i^{-1} \otimes x_{i,-r}^+ + x_{i,-r}^+ \otimes 1 + \sum_{j=1}^{r-1} \psi_{i,-j}^- \otimes x_{i,-r+j}^+ \quad (r > 0).$$

(c) Modulo $U_\xi(\tilde{\mathfrak{g}})(X^-)^{(2)} \otimes U_\xi(\tilde{\mathfrak{g}})X^+ + U_\xi(\tilde{\mathfrak{g}})X^- \otimes U_\xi(\tilde{\mathfrak{g}})X^+(i)$, we have

$$\Delta(x_{i,r}^-) = x_{i,r}^- \otimes k_i + 1 \otimes x_{i,r}^- + \sum_{j=1}^{r-1} x_{i,r-j}^- \otimes \psi_{i,j}^+ \quad (r > 0),$$

$$\Delta(x_{i,-r}^-) = x_{i,-r}^- \otimes k_i^{-1} + 1 \otimes x_{i,-r}^- + \sum_{j=1}^{r} x_{i,-r+j}^- \otimes \psi_{i,-j}^- \quad (r \geq 0).$$

(d) Modulo $U_\xi(\tilde{\mathfrak{g}})X^- \otimes U_\xi(\tilde{\mathfrak{g}})X^+$, we have

$$\Delta(h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r},$$

$$\Delta(\Lambda_{i,\pm r}) = \sum_{j=0}^{r} \Lambda_{i,\pm(r-j)} \otimes \Lambda_{i,\pm j},$$

$$\Delta(\psi_{i,\pm r}^{\pm}) = \sum_{j=0}^{r} \psi_{i,\pm(r-j)}^{\pm} \otimes \psi_{i,\pm j}^{\pm},$$

for every $r \in \mathbb{Z}_{>0}$.

If V is a representation of a Hopf algebra H, its dual space V^* is equipped with the structure of a representation by using the antipode. We shall not need the precise expression for the antipode of $U_q(\tilde{\mathfrak{g}})$.

Given $i \in I$ and $s \in \mathbb{Z}$, define

$$X_{i,s}^-(u) = \sum_{r \geq 1} x_{i,r+s}^- u^r. \tag{1.10}$$

Lemma 1.7. For every $i \in I$ and $s \in \mathbb{Z}$, we have

$$(x_{i,-s}^+)^{(l)}(x_{i,s+1}^-)^{(k)} = (-1)^l \left((X_{i,s}^-(u))^{(k-l)} \Lambda_i^+(u) \right)_k \mod U_{\mathbb{A}}(\tilde{\mathfrak{g}}) U_{\mathbb{A}}(\tilde{\mathfrak{n}}^+)^0.$$

Here $(X_{i,s}^-(u))^{(k-l)}$ is understood to be zero if $k < l$ and the subindex k on the right hand side means the coefficient of u^k in the given power series. Also, $U_{\mathbb{A}}(\tilde{\mathfrak{n}}^+)^0 = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\tilde{\mathfrak{n}}^+)^0$.

Proof. For $s \in \{0, -1\}$ this was proved in [19, §5]. The general case follows by applying the algebra automorphism determined by the assignment $(x_{i,r}^{\pm})^{(k)} \mapsto (x_{i,r\pm s}^{\pm})^{(k)}$. $\square$

The proofs of the next two lemmas can be found in [38].

Lemma 1.8. Let $m = m_0 + lm_1,\ r = r_0 + lr_1$ where $m, r \in \mathbb{Z},\ m \geq r,\ 0 \leq m_0, r_0 \leq l-1$.

(a) $\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]_{\zeta_i} = m_1$ and, if $m_1, r_1 \geq 0$, then $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_{\zeta_i} = \left[\begin{smallmatrix} m_0 \\ r_0 \end{smallmatrix}\right]_{\zeta_i} \binom{m_1}{r_1}$.

(b) If $m' \in \mathbb{Z}$ satisfies $\zeta_i^{m'} = \zeta_i^m$ and $\left[\begin{smallmatrix} m' \\ l \end{smallmatrix}\right]_{\zeta_i} = \left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]_{\zeta_i}$, then $m' = m$. $\square$

Lemma 1.9. The following identities hold in $U_\zeta(\tilde{\mathfrak{g}})$:

(a) $(x_{i,r}^{\pm})^{(m)} = (x_{i,r}^{\pm})^{(m_0)} \dfrac{((x_{i,r}^{\pm})^{(l)})^{m_1}}{m_1!}$ for all $i \in I, m, r \in \mathbb{Z}, m \geq 0$ such that $m = m_1 l + m_0$ with $0 \leq m_0 < l$.

(b) $\left[\begin{smallmatrix} k_i \\ m+l \end{smallmatrix}\right] = \left[\begin{smallmatrix} k_i \\ m \end{smallmatrix}\right] \left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} k_i \\ lr \end{smallmatrix}\right] = \dfrac{1}{r!} \displaystyle\prod_{j=0}^{r-1} \left(\left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right] - j \right)$ for all $i \in I, m, r \in \mathbb{Z}_{>0}$.

(c) $k_i^{2l} = 1$ and k_i^l is in the center of $U_\zeta(\mathfrak{g})$.

In particular, $U_\zeta(\tilde{\mathfrak{g}})$ is generated by $x_{i,r}^{\pm}$ and $(x_{i,r}^{\pm})^{(l)}$ with $i \in I, r \in \mathbb{Z}$, and $U_\zeta(\tilde{\mathfrak{h}})$ is generated by the elements $k_i, \left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right], \Lambda_{i,r}$ with $i \in I, r \in \mathbb{Z}$. $\square$

Lemma 1.10. The elements $\tilde{h}_{i,s} := \frac{h_{i,s}}{[s]_i}$ belong to $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$. Moreover, the image of $\tilde{h}_{i,s}$ in $U_\zeta(\tilde{\mathfrak{g}})$ is nonzero. In particular, $U_\zeta(\tilde{\mathfrak{h}})$ is generated by $\{k_i, \left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right], \tilde{h}_{i,s} : i \in I, s \in \mathbb{Z}_{>0}\}$ as a $\mathbb{C}$-algebra (with identity).

Proof. We have

$$-\sum_{s=1}^{\infty} \tilde{h}_{i,\pm s} u^s = \ln \left(\sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r \right) = \sum_{t=0}^{\infty} \frac{(-1)^t}{t+1} \left(\sum_{r=1}^{\infty} \Lambda_{i,\pm r} u^r \right)^{t+1}. \tag{1.11}$$

It follows that

$$\tilde{h}_{i,\pm s} = \sum_{0 \leq r \leq s} b_{i,s,r} \tilde{\Lambda}_{i,r}^{\pm}$$

with $b_{i,s,r} = -\frac{(-1)^r}{r+1} \in \mathbb{Q}$ and $\tilde{\Lambda}_{i,r}^{\pm} = \sum \Lambda_{i,\pm r_1} \cdots \Lambda_{i,\pm r_k}$ where $r_1 + \cdots + r_k = s, 1 \leq k \leq r+1, 1 \leq r_j \leq s$. Clearly the lemma now easily follows. $\square$

Let $U_\xi^{\text{fin}}(\tilde{\mathfrak{g}})$ be the subalgebra of $U_\xi(\tilde{\mathfrak{g}})$ generated by $U_\xi(\tilde{\mathfrak{h}})$ and $x_{i,r}^{\pm}, i \in I, r \in \mathbb{Z}$. Define $U_\xi^{\text{fin}}(\mathfrak{g})$ similarly. If ξ is not a root of unity, $U_\xi^{\text{fin}}(\tilde{\mathfrak{g}}) = U_\xi(\tilde{\mathfrak{g}})$ and $U_\xi^{\text{fin}}(\mathfrak{g}) = U_\xi(\mathfrak{g})$, but for $\xi = \zeta$ they are proper subalgebras.

Remark. The algebras $U_\zeta^{\text{fin}}(\mathfrak{g})$ and $U_\zeta^{\text{fin}}(\tilde{\mathfrak{g}})$ are enlargements (in the Cartan part only) of the so-called small quantum groups and small quantum affine algebra (see [25, §2.4] for instance).

1.4. **Braid group action and some identities.** In this section we state some identities that will be needed later. Given a $\mathbb{C}(q)$-associative algebra $A, x, y \in A$, and $p \in \mathbb{C}(q)$, define

$$[x, y]_p = xy - pyx. \tag{1.12}$$

When $p = 1$ this coincides with the usual commutator of associative algebras and the subindex 1 will be omitted.

The braid group $\mathcal{B}$ associated to $\mathfrak{g}$ is the multiplicative group generated by elements $T_i, i \in I$, with defining relations:

$$\begin{aligned}
T_i T_j &= T_j T_i, && \text{if } c_{ij} = 0, \\
T_i T_j T_i &= T_j T_i T_j, && \text{if } c_{ij} c_{ji} = 1, \\
(T_i T_j)^2 &= (T_j T_i)^2, && \text{if } c_{ij} c_{ji} = 2, \\
(T_i T_j)^3 &= (T_j T_i)^3, && \text{if } c_{ij} c_{ji} = 3.
\end{aligned}$$

The following proposition is well known.

Proposition 1.11. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$. The assignment

$$w \mapsto T_w = T_{i_1} \cdots T_{i_l}$$

does not depend on the choice of a reduced expression for w. $\square$

Let $\Omega, \Psi : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the $\mathbb{C}$-algebra anti-automorphisms defined by

$$\Omega x_i^\pm = x_i^\mp, \Omega k_i = k_i^{-1}, \Omega q = q^{-1} \quad \text{and} \quad \Psi x_i^\pm = x_i^\pm, \Psi k_i = k_i^{-1}, \Psi q = q.$$

The next theorem is proved by straightforward computations (see [39, §3]).

Theorem 1.12. For every $i \in I$, there exists a unique algebra automorphism T_i of $U_q(\mathfrak{g})$ such that $T_i x_i^+ = -x_i^- k_i, T_i x_i^- = -k_i^{-1} x_i^+, T_i k_j = k_j k_i^{-c_{ij}}$, and, for $j \neq i$,

$$T_i x_j^+ = \sum_{k=0}^{-c_{ij}} (-1)^k q_i^{c_{ij}+k} (x_i^+)^{(k)} x_j^+ (x_i^+)^{(-c_{ij}-k)}, \quad T_i x_j^- = \sum_{k=0}^{-c_{ij}} (-1)^k q_i^{-c_{ij}-k} (x_i^-)^{(-c_{ij}-k)} x_j^- (x_i^-)^{(k)}.$$

Moreover, the above defines a representation of $\mathcal{B}$ on $U_q(\mathfrak{g})$, T_i commutes with Ω, and $T_i^{-1} = \Psi T_i \Psi$. $\square$

Given $i, j \in I$ such that $c_{ji} = -1$, define the following elements of $U_q(\mathfrak{g})$:

$$X^\pm_{\alpha_i+\alpha_j} = [x_i^\pm, x_j^\pm]_{q_i^{\pm c_{ij}}}, \qquad X^\pm_{2\alpha_i+\alpha_j} = [x_i^\pm, X^\pm_{\alpha_i+\alpha_j}]_{q_i^{\pm c_{ij} \pm 2}}, \qquad X^\pm_{3\alpha_i+\alpha_j} = [x_i^\pm, X^\pm_{2\alpha_i+\alpha_j}]_{q_i^{\pm c_{ij} \pm 4}}$$

and

$$X^\pm_{3\alpha_i+2\alpha_j} = [X^\pm_{\alpha_i+\alpha_j}, X^\pm_{2\alpha_i+\alpha_j}]_{q_i^{\pm c_{ij} \pm 2}}.$$

Remark. Despite the ambiguity of the notation $X^\pm_{\alpha_i+\alpha_j}$ in the case $c_{ij} = c_{ji}$, this will not create confusion further on. Observe also that if $c_{ji} = -1$ and $c_{ij} < -1$, then $d_i = 1$ and, hence, $q_i = q$.

The next lemma is proved by a straightforward but tedious computation (cf.[39, §5]).

Lemma 1.13. Let $i, j \in I$ be such that $c_{ji} = -1$. Then, $T_j x_i^\pm = -X^\pm_{\alpha_i+\alpha_j} = -[x_i^\pm, x_j^\pm]_{q_i^{\pm c_{ij}}}$,

$$T_j T_i x_j^\pm = \begin{cases} x_i^\pm, & \text{if } c_{ij} = -1, \\ [2]_q^{-1} X^\pm_{2\alpha_i+\alpha_j}, & \text{if } c_{ij} = -2, \\ -(q[3]_q!)^{-1} X^\pm_{3\alpha_i+2\alpha_j}, & \text{if } c_{ij} = -3, \end{cases} \qquad T_j T_i T_j x_i^\pm = \begin{cases} x_i^\pm, & \text{if } c_{ij} = -2, \\ [2]_q^{-1} X^\pm_{2\alpha_i+\alpha_j}, & \text{if } c_{ij} = -3, \end{cases},$$

and, if $c_{ij} = -3$, then $T_j T_i T_j T_i x_j^\pm = ([3]_q!)^{-1} X^\pm_{3\alpha_i+\alpha_j}$ and $T_j T_i T_j T_i T_j x_i^\pm = x_i^\pm$. $\square$

Consider also the $\mathbb{C}(q)$-algebra isomorphism $\Phi : U_q(\mathfrak{n}^+) \to U_q(\mathfrak{n}^-), x_i^+ \mapsto x_i^-, i \in I$.

Definition 1.14. Let $i, j \in I$ be such that $c_{ji} = -1$ and let R^+_{ij} be the set of positive roots of the subalgebra of $\mathfrak{g}$ generated by $x_i^\pm$ and $x_j^\pm$. For $\alpha \in R^+_{ij}$ which is not a simple root, let $x^\pm_\alpha$ be the element on the left-hand side of the equalities in Lemma 1.13 which corresponds to $X^\pm_\alpha$. For $\alpha = \alpha_i$, $i \in I$, set $x^\pm_\alpha = x^\pm_i$. Define also $\tilde{x}^-_\alpha = \Phi(x^+_\alpha), \alpha \in R^+_{ij}$.

The first set of formulas in the next Lemma is obtained from [39, §5.3 and §5.4] by applying the anti-automorphism Ω while the second is obtained by applying Φ.

Lemma 1.15. Let $i,j\in I$ be such that $c_{ji}=-1$. Then,

$$
\begin{aligned}
(x_j^-)^{(l)}(x_i^-)^{(k)} &= \sum_{m\geq 0} q^{f_1(m)}(x_i^-)^{(k-m)}(x_{\alpha_i+\alpha_j}^-)^{(m)}(x_j^-)^{(l-m)}, && \text{if } c_{ij}=-1,\\
&= \sum_{s,t\geq 0} q^{f_2(s,t)}(x_i^-)^{(k-s-2t)}(x_{2\alpha_i+\alpha_j}^-)^{(t)}(x_{\alpha_i+\alpha_j}^-)^{(s)}(x_j^-)^{(l-s-t)}, && \text{if } c_{ij}=-2,\\
&= \sum_{m,r,s,t\geq 0} q^{f_3(m,r,s,t)}(x_i^-)^{(k-m-3r-2s-3t)}(x_{3\alpha_i+\alpha_j}^-)^{(t)}(x_{2\alpha_i+\alpha_j}^-)^{(s)}\times\\
&\qquad\qquad \times (x_{3\alpha_i+2\alpha_j}^-)^{(r)}(x_{\alpha_i+\alpha_j}^-)^{(m)}(x_j^-)^{(l-m-2r-s-t)}, && \text{if } c_{ij}=-3,
\end{aligned}
$$

$$
\begin{aligned}
(x_i^-)^{(k)}(x_j^-)^{(l)} &= \sum_{m\geq 0} q^{-f_1(m)}(x_j^-)^{(l-m)}(\tilde{x}_{\alpha_i+\alpha_j}^-)^{(m)}(x_i^-)^{(k-m)}, && \text{if } c_{ij}=-1,\\
&= \sum_{s,t\geq 0} q^{-f_2(s,t)}(x_j^-)^{(l-s-t)}(\tilde{x}_{\alpha_i+\alpha_j}^-)^{(s)}(\tilde{x}_{2\alpha_i+\alpha_j}^-)^{(t)}(x_i^-)^{(k-s-2t)}, && \text{if } c_{ij}=-2,\\
&= \sum_{m,r,s,t\geq 0} q^{-f_3(m,r,s,t)}(x_j^-)^{(l-m-2r-s-t)}(\tilde{x}_{\alpha_i+\alpha_j}^-)^{(m)}(\tilde{x}_{3\alpha_i+2\alpha_j}^-)^{(r)}(\tilde{x}_{2\alpha_i+\alpha_j}^-)^{(s)}\times\\
&\qquad\qquad \times (\tilde{x}_{3\alpha_i+\alpha_j}^-)^{(t)}(x_i^-)^{(k-m-3r-2s-3t)}, && \text{if } c_{ij}=-3,
\end{aligned}
$$

for some $f_1:\mathbb{Z}_{\geq 0}\to\mathbb{Z}$, $f_2:\mathbb{Z}_{\geq 0}^2\to\mathbb{Z}$, $f_3:\mathbb{Z}_{\geq 0}^4\to\mathbb{Z}$. In particular, $(x_\alpha^-)^{(k)},(\tilde{x}_\alpha^-)^{(k)}\in U_{\mathbb{A}}(\mathfrak{n}^-)$ for every $\alpha\in R_{ij}^+$ and all $k\in\mathbb{Z}_{\geq 0}$. □

The following is a particular case of [39, Proposition 5.7].

Theorem 1.16. Let $\mathfrak{g}$ have rank 2 and fix an ordering on R^+. Then, the corresponding ordered products of the elements $(x_\alpha^\pm)^{(k)},\alpha\in R^+,k\in\mathbb{Z}_{>0}$, form an $\mathbb{A}$-basis of $U_{\mathbb{A}}(\mathfrak{n}^\pm)$. □

Corollary 1.17. Let $\mathfrak{g}$ have rank 2 and fix an ordering on R^+. Then, the corresponding ordered products of the elements $(\tilde{x}_\alpha^-)^{(k)},\alpha\in R^+,k\in\mathbb{Z}_{>0}$, form an $\mathbb{A}$-basis of $U_{\mathbb{A}}(\mathfrak{n}^-)$. □

The proof the next lemma is straightforward.

Lemma 1.18. Let $i,j\in I$ be distinct, $r,s\in\mathbb{Z}$ and U_{ij}^{rs} be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $x_{i,r}^-$ and $x_{j,s}^-$. Then, the assignments $x_{i,r}^-\mapsto x_i^-, x_{j,s}^-\mapsto x_j^-$ extend uniquely to a monomorphism of algebras $U_{i,j}^{rs}\xrightarrow{\varphi_{i,j}^{r,s}} U_q(\mathfrak{n}^-)$. In particular, if $n=2$, $\varphi_{i,j}^{r,s}$ is an isomorphism. □

Given $i,j\in I$ such that $c_{ji}=-1$, $\alpha\in R_{ij}^+$, and $r,s\in\mathbb{Z}$, define

$$\gamma_\alpha^{r,s}=(\varphi_{i,j}^{r,s})^{-1}(x_\alpha^-)\qquad\text{and}\qquad \widetilde{\gamma}_\alpha^{r,s}=(\varphi_{i,j}^{r,s})^{-1}(\tilde{x}_\alpha^-). \tag{1.13}$$

It follows from the previous lemmas that $(\gamma_\alpha^{r,s})^{(k)},(\tilde{\gamma}_\alpha^{r,s})^{(k)}\in U_{\mathbb{A}}(\tilde{\mathfrak{n}}^-)$ for all $\alpha\in R_{ij}^+$ and all $r,s\in\mathbb{Z}$.

Moreover, by Lemma 1.13 and Theorem 1.16,

$$
\begin{gathered}
\gamma_{\alpha_i+\alpha_j}^{r,s}=-[x_{i,r}^-,x_{j,s}^-]_{q_i^{-c_{ij}}},\qquad \gamma_{2\alpha_i+\alpha_j}^{r,s}=-[2]_q^{-1}[x_{i,r}^-,\gamma_{\alpha_i+\alpha_j}^{r,s}]_{q^{-c_{ij}-2}},\\
\gamma_{3\alpha_i+\alpha_j}^{r,s}=[3]_q^{-1}[x_{i,r}^-,\gamma_{2\alpha_i+\alpha_j}^{r,s}]_{q^{-1}},\qquad \gamma_{3\alpha_i+2\alpha_j}^{r,s}=(q[3]_q)^{-1}[\gamma_{\alpha_i+\alpha_j}^{r,s},\gamma_{2\alpha_i+\alpha_j}^{r,s}]_q.
\end{gathered}\tag{1.14}
$$

Similar formulas for $\widetilde{\gamma}_\alpha^{r,s}$ are obtained by replacing q with q^{-1} in (1.14).

Lemma 1.19. Let $i,j\in I$ be such that $c_{ji}=-1$ and $r,s\in\mathbb{Z}$. Then,

$$\gamma^{r,s}_{\alpha_i+\alpha_j}=q_i^{-c_{ij}}\widetilde\gamma^{r-1,s+1}_{\alpha_i+\alpha_j},\quad \gamma^{r,s}_{2\alpha_i+\alpha_j}=q^{-2c_{ij}-2}\widetilde\gamma^{r-1,s+2}_{2\alpha_i+\alpha_j},\quad\text{and}\quad \gamma^{r,s}_{3\alpha_i+\alpha_j}=q^{3}\widetilde\gamma^{r-1,s+3}_{3\alpha_i+\alpha_j}.$$

Proof. Let us first record the following identity which holds for any $i\in I$:

$$x^-_{i,r}x^-_{i,r-1}=q_i^{-2}x^-_{i,r-1}x^-_{i,r}. \tag{1.15}$$

This and the first identity of the lemma are immediate from the defining relation

$$x^-_{i,r}x^-_{j,s}-q_i^{-c_{ij}}x^-_{j,s}x^-_{i,r}=q_i^{-c_{ij}}x^-_{i,r-1}x^-_{j,s+1}-x^-_{j,s+1}x^-_{i,r-1}.$$

As for the second identity we proceed as follows. By the first identity and (1.14), we have

$$-[2]_q\gamma^{r,s}_{2\alpha_i+\alpha_j}=[x^-_{i,r},q^{-c_{ij}}\widetilde\gamma^{r-1,s+1}_{\alpha_i+\alpha_j}]_{q^{-c_{ij}-2}}.$$

Then

$$\begin{aligned}[2]_q\gamma^{r,s}_{2\alpha_i+\alpha_j}&=x^-_{i,r}(q^{-c_{ij}}x^-_{i,r-1}x^-_{j,s+1}-x^-_{j,s+1}x^-_{i,r-1})-q^{-c_{ij}-2}(q^{-c_{ij}}x^-_{i,r-1}x^-_{j,s+1}-x^-_{j,s+1}x^-_{i,r-1})x^-_{i,r}\\&=q^{-c_{ij}-2}x^-_{i,r-1}(-\gamma^{r,s+1}_{\alpha_i+\alpha_j})-(-\gamma^{r,s+1}_{\alpha_i+\alpha_j})x^-_{i,r-1}=-q^{-c_{ij}-2}[x^-_{i,r-1},q^{-c_{ij}}\widetilde\gamma^{r-1,s+2}_{\alpha_i+\alpha_j}]_{q^{c_{ij}+2}}\\&=[2]_qq^{-2c_{ij}-2}\widetilde\gamma^{r-1,s+2}_{2\alpha_i+\alpha_j}.\end{aligned}$$

In the second equality above we used (1.15), while the first identity was used once more in the third equality. The last equality holds since $\widetilde\gamma^{r,s}_{2\alpha_i+\alpha_j}=-[2]^{-1}_{q^{-1}}[x^-_{i,r},\widetilde\gamma^{r,s}_{\alpha_i+\alpha_j}]_{q^{c_{ij}+2}}$ by (1.14) and $[2]_q=[2]_{q^{-1}}$.

The last identity is proved similarly and we omit the details. $\square$

Remark. The elements $\gamma^{r,s}_\alpha$ were first considered in [20] for simply laced $\mathfrak g$ and in [7] for $\mathfrak g$ of classical type. We warn the reader that there is a couple of typos in [7] regarding $\gamma^{r,s}_{2\alpha_i+\alpha_j}$ (denoted by $\gamma'_{r,s}(q)$ there): one in its definition and the other in the identity corresponding to the third identity of the above lemma.

2. Braid Group and ℓ-Lattices

2.1. **The ℓ-Lattices.** Given a field $\mathbb F$ consider the multiplicative group $\mathcal P_{\mathbb F}$ of n-tuples of rational functions $\boldsymbol\mu=(\boldsymbol\mu_1(u),\cdots,\boldsymbol\mu_n(u))$ with values in $\mathbb F$ such that $\boldsymbol\mu_i(0)=1$ for all $i\in I$. We shall often think of $\boldsymbol\mu_i(u)$ as a formal power series in u with coefficients in $\mathbb F$. Given $a\in\mathbb F^\times$ and $i\in I$, let $\boldsymbol\omega_{i,a}$ be defined by

$$(\boldsymbol\omega_{i,a})_j(u)=1-\delta_{i,j}au.$$

Clearly, if $\mathbb F$ is algebraically closed, $\mathcal P_{\mathbb F}$ is the free abelian group generated by these elements which are called fundamental ℓ-weights. It is also convenient to introduce elements $\boldsymbol\omega_{\lambda,a},\lambda\in P,a\in\mathbb F^\times$, defined by

$$\boldsymbol\omega_{\lambda,a}=\prod_{i\in I}(\boldsymbol\omega_{i,a})^{\lambda(h_i)}. \tag{2.1}$$

If $\mathbb F$ is algebraically closed, introduce the group homomorphism (weight map) $\mathrm{wt}:\mathcal P_{\mathbb F}\to P$ by setting $\mathrm{wt}(\boldsymbol\omega_{i,a})=\omega_i$, where ω_i is the i-th fundamental weight of $\mathfrak g$. Otherwise, let $\mathbb K$ be an algebraically closed extension of $\mathbb F$ and regard $\mathcal P_{\mathbb F}$ as a subgroup of $\mathcal P_{\mathbb K}$. Then, define the weight map on $\mathcal P_{\mathbb K}$ as above and on $\mathcal P_{\mathbb F}$ by restriction (this clearly does not depend on the choice of $\mathbb K$). Define the ℓ-weight lattice of $U_q(\tilde{\mathfrak g})$ to be $\mathcal P_q:=\mathcal P_{\mathbb C(q)}$. The submonoid $\mathcal P_q^+$ of $\mathcal P_q$ consisting of n-tuples of polynomials is called the set of dominant ℓ-weights of $U_q(\tilde{\mathfrak g})$.

Given $\boldsymbol\lambda\in\mathcal P_q^+$ with $\boldsymbol\lambda_i(u)=\prod_j(1-a_{i,j}u)$, where $a_{i,j}$ belong to some algebraic closure of $\mathbb C(q)$, let $\boldsymbol\lambda^-\in\mathcal P_q^+$ be defined by $\boldsymbol\lambda_i^-(u)=\prod_j(1-a_{i,j}^{-1}u)$. We will also use the notation $\boldsymbol\lambda^+=\boldsymbol\lambda$. Two elements

$\boldsymbol{\lambda}, \boldsymbol{\mu}$ of $\mathcal{P}_q^+$ are said to be relatively prime if $\boldsymbol{\lambda}_i(u)$ is relatively prime to $\boldsymbol{\mu}_j(u)$ in $\mathbb{C}(q)[u]$ for all $i, j \in I$. Every $\boldsymbol{\nu} \in \mathcal{P}_q$ can be uniquely written in the form

$$\boldsymbol{\nu} = \boldsymbol{\lambda}\boldsymbol{\mu}^{-1} \quad \text{with} \quad \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_q^+ \quad \text{relatively prime.} \tag{2.2}$$

Given $\boldsymbol{\nu} = \boldsymbol{\lambda}\boldsymbol{\mu}^{-1}$ as above, define a $\mathbb{C}(q)$-algebra homomorphism $\boldsymbol{\Psi}_{\boldsymbol{\nu}} : U_q(\tilde{\mathfrak{h}}) \to \mathbb{C}(q)$ by setting

$$\boldsymbol{\Psi}_{\boldsymbol{\nu}}(k_i^{\pm 1}) = q_i^{\pm \mathrm{wt}(\boldsymbol{\nu})(h_i)}, \qquad \sum_{r \geq 0} \boldsymbol{\Psi}_{\boldsymbol{\nu}}(\Lambda_{i,\pm r}) u^r = \frac{(\boldsymbol{\lambda}^{\pm})_i(u)}{(\boldsymbol{\mu}^{\pm})_i(u)} \tag{2.3}$$

where the division is that of formal power series in u.

The next proposition is easily checked.

Proposition 2.1. The map $\boldsymbol{\Psi} : \mathcal{P}_q \to (U_q(\tilde{\mathfrak{h}}))^*$ given by $\boldsymbol{\nu} \mapsto \boldsymbol{\Psi}_{\boldsymbol{\nu}}$ is injective. □

Let $\mathcal{P}$ be the subgroup of $\mathcal{P}_q$ generated by $\boldsymbol{\omega}_{i,a}$ for all $i \in I$ and all $a \in \mathbb{C}^\times$ (equivalently, $\mathcal{P} = \mathcal{P}_{\mathbb{C}}$). Set also $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{P}_q^+$. From now on we will identify $\mathcal{P}_q$ with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under $\boldsymbol{\Psi}$. Similarly, given $\xi \in \mathbb{C}^\times$, $\mathcal{P}$ can be identified with a subset of $U_\xi(\tilde{\mathfrak{h}})^*$ by using the same expression for $\boldsymbol{\Psi}_{\boldsymbol{\nu}}(\Lambda_{i,\pm r})$ and

$$\boldsymbol{\Psi}_{\boldsymbol{\nu}}(k_i^{\pm 1}) = \xi_i^{\pm \mathrm{wt}(\boldsymbol{\nu})(h_i)} \qquad \text{and} \qquad \boldsymbol{\Psi}_{\nu}\left(\left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \mathrm{wt}(\boldsymbol{\nu})(h_i) \\ l \end{smallmatrix}\right]_{\xi_i}.$$

Observe also that every element $\boldsymbol{\lambda} \in \mathcal{P}_q$ such that $\boldsymbol{\lambda}_i(u)$ splits in $\mathbb{C}(q)[u]$ for all $i \in I$ can be uniquely decomposed as

$$\boldsymbol{\lambda} = \prod_j \boldsymbol{\omega}_{\lambda_j, a_j} \quad \text{for some} \quad \lambda_j \in P \quad \text{and} \quad a_i \neq a_j. \tag{2.4}$$

For notational convenience, henceforth we set $\mathcal{P}_\xi = \mathcal{P}$ when $\xi \neq q$. It will be convenient to introduce the following notation. Given $i \in I, a \in \mathbb{C}(\xi)^\times, r \in \mathbb{Z}_{\geq 0}$, define

$$f_{i,a,r}(u) = \prod_{j=0}^{r-1} (1 - a\xi_i^{r-1-2j} u) \qquad \text{and} \qquad \boldsymbol{\omega}_{i,a,r} = \prod_{j=0}^{r-1} \boldsymbol{\omega}_{i, a\xi_i^{r-1-2j}}. \tag{2.5}$$

Even though the dependence on ξ is missing in the notation above, this will not create a confusion. Suppose ξ has infinite order and observe that if $f(u) \in \mathbb{C}(\xi)[u]$ splits in $\mathbb{C}(\xi)[u]$ and $f(0) = 1$, then there exist unique $m \in \mathbb{Z}_{\geq 0}$, $a_1, \dots, a_m \in \mathbb{C}(\xi)^\times$, and $r_1, \dots, r_m \in \mathbb{Z}_{\geq 1}$ such that

$$f(u) = \prod_{k=1}^{m} f_{i,a_k,r_k}(u) \qquad \text{with} \qquad \frac{a_k}{a_j} \neq \xi_i^{\pm(r_k + r_j - 2p)} \quad \text{for} \quad k \neq j, \ 0 \leq p < \min\{r_k, r_j\}. \tag{2.6}$$

In particular, given $\boldsymbol{\lambda} \in \mathcal{P}_\xi^+$ such that $\boldsymbol{\lambda}_i(u)$ splits in $\mathbb{C}(\xi)[u]$ for all $i \in I$, there exist unique $m_i \in Z_{\geq 0}, a_{i,k} \in \mathbb{C}(\xi)^\times$, and $r_{i,k} \in \mathbb{Z}_{\geq 1}$ such that

$$\begin{gathered} \boldsymbol{\lambda} = \prod_{i \in I} \prod_{k=1}^{m_i} \boldsymbol{\omega}_{i, a_{i,k}, r_{i,k}} \qquad \text{with} \\ \frac{a_{i,k}}{a_{i,j}} \neq \xi_i^{\pm(r_{i,k} + r_{i,j} - 2p)} \quad \text{for all} \quad i \in I, k \neq j, 0 \leq p < \min\{r_{i,k}, r_{i,j}\}. \end{gathered} \tag{2.7}$$

The factorization (2.6) is called the ξ_i-factorization of f and the factors f_{i,a_k,r_k} are called the ξ_i-factors of f. Two polynomials are said to be in ξ_i-general position if the set of ξ_i-factors of their product is the union of the corresponding sets of ξ_i-factors (counted with multiplicities). Let $f, g \in \mathbb{C}(q)[u]$ be

ξ_i-factorizable with ξ_i-factors $\{f_{i,a_k,r_k} : k = 1, \dots, m_1\}$ and $\{f_{i,b_j,s_j} : j = 1, \dots, m_2\}$, respectively. An ordered pair (f, g) is said to be in ξ_i-resonant order if

$$\frac{a_k}{b_j} \neq \xi_i^{-(r_k+s_j-2p)} \quad \text{for all} \quad k, j \quad \text{and all} \quad 0 \leq p < r_k. \tag{2.8}$$

We shall also define the notion of weak ξ_i-resonant order by asking that

$$\frac{a_k}{b_j} \neq \xi_i^{-(r_k+s_j-2p)} \quad \text{for all} \quad k, j \quad \text{and all} \quad 0 \leq p < \min\{r_k, s_j\}. \tag{2.9}$$

Clearly, if (f, g) is in ξ_i-resonant order it is in weak ξ_i-resonant order as well. Similarly, an ordered collection $(f_1, f_2, \cdots, f_m)$ of polynomials is said to be in (weak) ξ_i-resonant order if (f_j, f_k) is in (weak) ξ_i-resonant order for every $j < k$. One also defines the concept of general position for a finite collection of polynomials by requiring that they are pairwise in general position.

Remark. If $m_1 = m_2 = 1$, then either (f, g) or (g, f) is in ξ_i-resonant order. In general, both (f, g) and (g, f) may not be in weak ξ_i-resonant order. Also, given a finite collection of polynomials $f_1, \cdots, f_m$ whose ξ_i-factorizations have only one ξ_i-factor, then they can be arranged to be in ξ_i-resonant order. The concept of resonant order was introduced in [8], but with different terminology. Namely, the polynomial g was said to be in general position with respect to f if (2.8) is satisfied. The terminology "resonant" was used in [22]. Notice that f and g are in general position iff both (f, g) and (g, f) are in weak ξ_i-resonant order.

Let $\phi_l : \mathbb{C}(q)[u] \to \mathbb{C}(q)[u]$ be the $\mathbb{C}(q)$-algebra homomorphism such that $\phi_l(u) = u^l$. We also denote by ϕ_l the induced group homomorphism $\mathcal{P}_q \to \mathcal{P}_q$. Observe that, if $a, b \in \mathbb{C}(q)$ are such that $b^l = a$, then

$$\phi_l(1 - au) = 1 - au^l = \prod_{j=0}^{l-1}(1 - b\zeta^j u) \tag{2.10}$$

and that every $\boldsymbol{\lambda} \in \mathcal{P}^+$ admits a unique decomposition of the form

$$\boldsymbol{\lambda}(u) = \boldsymbol{\lambda}'\phi_l(\boldsymbol{\lambda}'') \qquad \text{with} \qquad \boldsymbol{\lambda}', \boldsymbol{\lambda}'' \in \mathcal{P}^+, \tag{2.11}$$

where $\boldsymbol{\lambda}'$ satisfies

$$\frac{\boldsymbol{\lambda}_i'}{1 - au^l} \notin \mathcal{P}^+ \qquad \text{for all} \qquad i \in I, \ a \in \mathbb{C}^\times.$$

Set $\mathcal{P}_l^+ = \{\boldsymbol{\lambda} \in \mathcal{P}^+ : \boldsymbol{\lambda} = \boldsymbol{\lambda}'\}$.

Denote by $\mathcal{P}_\mathbb{A}^+$ be the subset of $\mathcal{P}_q$ consisting of n-tuples of polynomials with coefficients in $\mathbb{A}$ and $\mathcal{P}_\mathbb{A}$ the corresponding subgroup. Let $\mathcal{P}_\mathbb{A}^{++}$ be the submonoid of $\mathcal{P}_\mathbb{A}^+$ consisting of n-tuples of polynomials with invertible leading coefficients and $\mathcal{P}_\mathbb{A}^s = \{\boldsymbol{\lambda} \in \mathcal{P}_q^+ :$ the roots of $\boldsymbol{\lambda}_i(u)$ lie in $\mathbb{A}^\times$ for all $i \in I\} \subseteq \mathcal{P}_\mathbb{A}^{++}$. We may also use the terminology $\boldsymbol{\lambda}$ splits in $\mathbb{A}^\times$ to mean $\boldsymbol{\lambda} \in \mathcal{P}_\mathbb{A}^s$. Notice that, if $\boldsymbol{\lambda} \in \mathcal{P}_\mathbb{A}^{++}$, then $\boldsymbol{\lambda}_i(u)$ splits in $\mathbb{A}$ for all $i \in I$ iff $\boldsymbol{\lambda}$ splits in $\mathbb{A}^\times$.

Given $f \in \mathbb{A}(u)$, let $\bar{f}$ be the element of $\mathbb{C}(u)$ obtained from f by evaluating q at a given $\xi \in \mathbb{C}^\times$. Similarly, given $\boldsymbol{\lambda} \in \mathcal{P}_\mathbb{A}$, let $\overline{\boldsymbol{\lambda}}$ be the element of $\mathcal{P}$ obtained from $\boldsymbol{\lambda}$ by evaluating q at ξ.

For the remainder of the section we assume $\xi = \zeta$. In this case we have

$$\overline{f_{i,a,l}(u)} = 1 - a^l u^l \qquad \text{for all} \qquad i \in I, a \in \mathbb{C}^\times. \tag{2.12}$$

It is now easy to see that, for every $i \in I$ and $f(u) \in \mathbb{C}[u]$ satisfying $f(0) = 1$, there exist unique $m_0, m_1 \in \mathbb{Z}_{\geq 0}, a_1, \dots, a_{m_0}, b_1, \dots, b_{m_1} \in \mathbb{C}^\times$, and $r_1, \dots, r_{m_0}, s_1, \dots, s_{m_1} \in \mathbb{Z}_{>0}$, such that

$$f(u) = \left(\prod_{k=1}^{m_0} f_{i,a_k,r_k}(u)\right)\left(\prod_{k=1}^{m_1}(1 - b_k u^l)^{s_k}\right) \qquad \text{with} \qquad r_k < l,$$

(2.13)

$$\frac{a_k}{a_j} \neq \zeta_i^{\pm(r_k+r_j-2p)} \quad \text{and} \quad b_k \neq b_j \quad \text{for} \quad 0 \leq p < \min\{r_k, r_j\} \quad \text{and} \quad k \neq j.$$

In particular, given $\boldsymbol{\lambda} \in \mathcal{P}^+$, there exist unique $m, m_i, \in \mathbb{Z}_{\geq 0}, a_{i,1}, \dots, a_{i,m_i}, b_1, \dots, b_m \in \mathbb{C}^\times, \lambda_1, \dots, \lambda_m \in P^+ \backslash \{0\}$, and $r_{i,1}, \dots, r_{i,m_i} \in \mathbb{Z}_{>0}$ such that

$$\boldsymbol{\lambda} = \left(\prod_{i \in I}\prod_{k=1}^{m_i} \boldsymbol{\omega}_{i,a_{i,k},r_{i,k}}(u)\right)\left(\prod_{k=1}^{m} \phi_l(\boldsymbol{\omega}_{\lambda_k,b_k})\right) \qquad \text{with} \qquad r_{i,k} < l,$$

(2.14)

$$\frac{a_{i,k}}{a_{i,j}} \neq \zeta_i^{\pm(r_{i,k}+r_{i,j}-2p)} \quad \text{and} \quad b_k \neq b_j \quad \text{for all} \quad i \in I, k \neq j, 0 \leq p < \min\{r_{i,k}, r_{i,j}\}.$$

The factorization (2.13) is called the ζ_i-factorization of f. The factors f_{i,a_k,r_k} are called the quantum ζ_i-factors of f while the factors $(1 - b_k u^l)^{s_k}$ are called the Frobenius ζ_i-factors. We will refer to both kinds of factors simply by the ζ_i-factors of f. Two polynomials are said to be in ζ_i-general position if the ζ_i-factors of their product are the union of their ζ_i-factors (counted with multiplicities). Let $f, g \in \mathbb{C}[u]$ have ζ_i-factors $\{f_{i,a_k,r_k}, (1-b_j u^l)^{s_j} : k = 1, \dots, m_0, j = 1, \dots, m_1\}$ and $\{f_{i,a'_k,r'_k}, (1-b'_j u^l)^{s'_j} : k = 1, \dots, m'_0, j = 1, \dots, m'_1\}$, respectively. We say that the pair (f, g) is in ζ_i-resonant order if

$$\frac{a_k}{a'_j} \neq \zeta_i^{-(r_k+r'_j-2p)} \quad \text{and} \quad b_k \neq b'_j \quad \text{for all} \quad k, j \quad \text{and all} \quad 0 \leq p < r_k. \tag{2.15}$$

The concepts of general position and (weak) ζ_i-resonant order for a finite ordered collection of polynomials is defined in the obvious way.

Remark. Suppose $l = 3$ and consider the polynomials $f_1(u) = 1 - u, f_2(u) = 1 - \zeta u$, and $f_3(u) = 1 - \zeta^2 u$. It is easy to see that f_1, f_2, f_3 cannot be arranged in a weak ζ-resonant order.

2.2. **Braid group action on the ℓ-weight lattices.** The following proposition was proved in [6, 8].

Proposition 2.2. The following formulas define an action of $\mathcal{B}$ on $\mathcal{P}_q$:

$$(T_i\boldsymbol{\mu})_i(u) = (\boldsymbol{\mu}_i(\xi_i^2 u))^{-1} \qquad \text{and} \qquad (T_i\boldsymbol{\mu})_j(u) = \boldsymbol{\mu}_j(u)\prod_{k=1}^{|c_{ji}|}\boldsymbol{\mu}_i(\xi^{d_i+|c_{ji}|+1-2k}u).$$

In particular, $\mathrm{wt}(T_w\boldsymbol{\mu}) = w\mathrm{wt}(\boldsymbol{\mu})$ for all $w \in \mathcal{W}$. Moreover, $T_i\boldsymbol{\mu} \in \mathcal{P}_{\mathbb{A}}$ for $\boldsymbol{\mu} \in \mathcal{P}_{\mathbb{A}}$. □

By composing the above action with the evaluation map $\epsilon_\xi, \xi \in \mathbb{C}^\times$, one gets an action of $\mathcal{B}$ on $\mathcal{P}$. Hence, if $\boldsymbol{\mu} \in \mathcal{P}$, there is an ambiguity in the notation $T_i\boldsymbol{\mu}$. It will always be clear from the context if we mean the "q-action" or the "ξ-action" for some $\xi \in \mathbb{C}^\times$.

Lemma 2.3. Suppose $i \in I$ and $w \in \mathcal{W}$ are such that $\ell(s_i w) = \ell(w) + 1$. Then, $(T_w\boldsymbol{\lambda})_i(u) \in \mathbb{C}(\xi)[u]$ for every $\boldsymbol{\lambda} \in \mathcal{P}_\xi^+$.

Proof. It follows from [8, Proposition 3.2]. Alternatively, see the the proof of Corollary 4.8 below. □

Recall that w_0 defines a Dynkin diagram automorphism such that $w_0 \cdot i = j$ iff $w_0\omega_i = -\omega_j$ for $i, j \in I$. Given $\boldsymbol{\lambda} \in \mathcal{P}_q^+$, denote by $\boldsymbol{\lambda}^*$ any of the three elements defined by the corresponding action of $\mathcal{B}$ given by the following expression

$$\boldsymbol{\lambda}^* = (T_{w_0}\boldsymbol{\lambda})^{-1}. \tag{2.16}$$

A straightforward but tedious computation working with a reduced expression for w_0 shows that

$$(\boldsymbol{\lambda}^*)_i(u) = \boldsymbol{\lambda}_{w_0 \cdot i}(\xi^{r^\vee h^\vee} u) \tag{2.17}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $r^\vee = \max\{c_{ij}c_{ji} : i, j \in I, i \neq j\}$ is the lacing number of $\mathfrak{g}$.

Fix a reduced expression for w_0, say $s_{i_N} \cdots s_{i_1}$. A pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of dominant ℓ-weights is said to be in (weak) ξ-resonant order (with respect to this reduced expression of w_0) if

(a) $(\boldsymbol{\lambda}_i(u), \boldsymbol{\mu}_i(u))$ is in (weak) ξ_i-resonant order for every $i \in I$,
(b) $((T_{i_{j-1}} \cdots T_{i_1}\boldsymbol{\lambda})_{i_j}(u), \boldsymbol{\mu}_{i_j}(u))$ is in (weak) ξ_{i_j}-resonant order for every $j = 1, \ldots, N$.

The concept of ξ-resonant order for m-tuples of elements of $\mathcal{P}_\xi^+$ is defined in the obvious way.

Remark. If ξ is not a root of unity, condition (a) above follows from condition (b) (see the proof of [8, Theorem 4.4]).

Lemma 2.4. Let $k \in \mathbb{Z}_{\geq 1}, i_j \in I, m_j \in \mathbb{Z}_{\geq 1}, a_j \in \mathbb{C}(q)^\times$ for $j = 1, \ldots, k$. Then:

(a) If $\frac{a_j}{a_m} \notin q^{\mathbb{Z}_{>0}}$ for $j > m$, then $(\boldsymbol{\omega}_{i_1,a_1,m_1}, \cdots, \boldsymbol{\omega}_{i_k,a_k,m_k})$ is in q-resonant order.
(b) There exists $\sigma \in S_k$ (independent of $i_1, \cdots, i_k$) such that $(\boldsymbol{\omega}_{i_1,a_{\sigma(1)},m_1}, \cdots, \boldsymbol{\omega}_{i_k,a_{\sigma(k)},m_k})$ is in q-resonant order.

Proof. Part (a) is proved by a direct computation (see [8, §6]). Part (b) is easily established from part (a). □

Remark. Because of the closing remark of the previous subsection, there is no "sufficiently" general analogue of Lemma 2.4(b) in the root of unity context. The lack of a root of unity analogue of part (a) of Lemma 2.4 and, consequently also of part (b), will imply the corresponding lack of a root of unity analogue of Corollary 4.23 below.

We now use the braid group action to define the ℓ-analogue of simple roots. These objects were defined originally in [26, 25] (without using the braid group) and in [12]. Given $a \in \mathbb{C}(\xi)^\times$, define the simple ℓ-root $\boldsymbol{\alpha}_{i,a}$ by

$$\boldsymbol{\alpha}_{i,a} := \boldsymbol{\omega}_{i,a}(T_i\boldsymbol{\omega}_{i,a})^{-1} = (\boldsymbol{\omega}_{i,a\xi_i,2})^{-1} \prod_{j \neq i} \boldsymbol{\omega}_{j,a\xi_i,-c_{j,i}}. \tag{2.18}$$

The subgroup of $\mathcal{P}_\xi$ generated by the simple ℓ-roots $\boldsymbol{\alpha}_{i,a}, i \in I, a \in \mathbb{C}(\xi)^\times$, is called the ℓ-root lattice of $U_\xi(\tilde{\mathfrak{g}})$ and will be denoted by $\mathcal{Q}_\xi$. Let also $\mathcal{Q}_\xi^+$ be the submonoid generated by the simple ℓ-roots. Quite clearly $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$. Define a partial order on $\mathcal{P}_\xi$ by

$$\boldsymbol{\mu} \leq \boldsymbol{\lambda} \qquad if \qquad \boldsymbol{\lambda}\boldsymbol{\mu}^{-1} \in \mathcal{Q}_\xi^+.$$

One easily checks the following proposition (cf. [12, §3.3]).

Proposition 2.5. The action of the braid group on $\mathcal{P}_\xi$ preserves $\mathcal{Q}_\xi$. □

2.3. **The group of elliptic characters.** This subsection will be used only in §4.6. Set

$$\widetilde{\Gamma}_\xi = \mathcal{P}_\xi / \mathcal{Q}_\xi. \tag{2.19}$$

The elements of the group $\widetilde{\Gamma}_\xi$ are called elliptic characters. The motivation for this name will be explained in the remark after Proposition 2.6 below. We denote by γ_ξ the canonical projection $\mathcal{P}_\xi \to \widetilde{\Gamma}_\xi$.

Define subsets $\mathcal{T}_k$ of $\mathbb{Z}_{\geq 0}, k = 1, 2, 3$, as follows.

$$\begin{array}{ll}
\mathcal{T}_1 = \{0, 2, \cdots, 2n\}, \quad \mathcal{T}_2 = \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } A_n, \\
\mathcal{T}_1 = \{0, 4n-2\}, \quad \mathcal{T}_2 = \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } B_n, \\
\mathcal{T}_1 = \{0, 2n+2\}, \quad \mathcal{T}_2 = \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } C_n, \\
\mathcal{T}_1 = \{0, 2, 2n-2, 2n\}, \quad \mathcal{T}_2 = \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } D_n, \\
\mathcal{T}_1 = \{0, 8, 16\}, \quad \mathcal{T}_2 = \{0, 2, 4, 12, 14, 16\}, \quad \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } E_6, \\
\mathcal{T}_1 = \{0, 18\}, \quad \mathcal{T}_2 = \{0, 2, 12, 14, 24, 26\}, \quad \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } E_7, \\
\mathcal{T}_1 = \{0, 30\}, \quad \mathcal{T}_2 = \{0, 20, 40\}, \quad \mathcal{T}_3 = \{0, 12, 24, 36, 48\}, & \text{if } \mathfrak{g} \text{ is of type } E_8, \\
\mathcal{T}_1 = \{0, 18\}, \quad \mathcal{T}_2 = \{0, 12, 24\}, \quad \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } F_4, \\
\mathcal{T}_1 = \{0, 12\}, \quad \mathcal{T}_2 = \{0, 8, 16\}, \quad \mathcal{T}_3 = \emptyset, & \text{if } \mathfrak{g} \text{ is of type } G_2.
\end{array}$$

Given $k \in \{1, 2, 3\}$ and $a \in \mathbb{C}(\xi)^\times$, define

$$\boldsymbol{\tau}_{\xi,k,a} = \prod_{r \in \mathcal{T}_k} \boldsymbol{\omega}_{i, a\xi^r} \tag{2.20}$$

if $\mathfrak{g}$ is not of type D_n with n even and where $\{i\} = I_\bullet$ (see Table 1). For $\mathfrak{g}$ of type D_n with n even define

$$\begin{gathered}
\boldsymbol{\tau}_{\xi,1,a} = \boldsymbol{\omega}_{n-1,a}\boldsymbol{\omega}_{n-1,a\xi^{2n-2}}, \qquad \boldsymbol{\tau}_{\xi,2,a} = \boldsymbol{\omega}_{n,a}\boldsymbol{\omega}_{n,a\xi^{2n-2}}, \\
\text{and} \\
\boldsymbol{\tau}_{\xi,3,a} = \boldsymbol{\omega}_{n-1,a}\boldsymbol{\omega}_{n-1,a\xi^2}\boldsymbol{\omega}_{n,a\xi^{2n-2}}\boldsymbol{\omega}_{n,a\xi^{2n}}.
\end{gathered} \tag{2.21}$$

Remark. We warn the reader that there were a couple of typos in [12, §A.2] in the definition of the sets corresponding to our $\mathcal{T}_k$. They have been corrected above.

Proposition 2.6. Assume $\mathfrak{g}$ is not of type D_n with n even. Then, the group $\widetilde{\Gamma}_\xi$ is isomorphic to the (additive) abelian group with generators $\{\chi_a : a \in \mathbb{C}(\xi)^\times\}$ and relations

$$\sum_{j \in \mathcal{T}_k} \chi_{a\xi^j} = 0,$$

for all $a \in \mathbb{C}(\xi)^\times$ and $k = 1, 2, 3$. If $\mathfrak{g}$ is of type D_n with n even, then $\widetilde{\Gamma}_\xi$ is isomorphic to the (additive) abelian group with generators $\{\chi_a^\pm : a \in \mathbb{C}(\xi)^\times\}$ and relations

$$\chi_a^\pm + \chi_{a\xi^{2n-2}}^\pm = 0, \quad \chi_a^- + \chi_{a\xi^2}^- + \chi_{a\xi^{2n-2}}^+ + \chi_{a\xi^{2n}}^+ = 0,$$

for all $a \in \mathbb{C}(\xi)^\times$.

Proof. The proposition was proved in [12] for $\xi = q$ and the proof is analogous for the other cases. The isomorphism is given by sending the image of $\boldsymbol{\omega}_{i,a}$ in $\widetilde{\Gamma}_\xi$ to the generator χ_a for $i \in I_\bullet$ (in the case $I_\bullet$ has two elements, the image of $\boldsymbol{\omega}_{n-1,a}$ is sent to χ_a^- while the image of $\boldsymbol{\omega}_{n,a}$ is sent to χ_a^+). ☐

Remark. Observe that the relations above imply $\chi_a = \chi_{a\xi^{2r^\vee h^\vee}}$ for every $a \in \mathbb{C}(\xi)^\times$ (and similarly for $\chi_a^\pm$). This is the motivation for the terminology elliptic characters. Namely, if $\xi \in \mathbb{C}^\times$ is not a root of unity, the parametrizing set for the distinct generators of the form χ_a is the elliptic curve $\mathbb{C}^\times/\xi^{2r^\vee h^\vee\mathbb{Z}}$. For $\xi = 1$, the elliptic behavior degenerates and the elements of $\widetilde{\Gamma}_1$ are usually called spectral characters. Notice that $\widetilde{\Gamma}_1$ is isomorphic to the group of functions with finite support from $\mathbb{C}^\times$ to P/Q (cf. [11]). If $\xi = \zeta$, the elliptic curve parametrizing the distinct generators χ_a depends on the greatest common divisor of $2r^\vee h^\vee$ and the order l of ζ. For instance, let $\mathfrak{g} = \mathfrak{sl}_2$ so that l is relatively prime to $2r^\vee h^\vee = 4$. One can check that we have $\chi_a = \chi_{a\zeta^m}$ for any $m \in \mathbb{Z}$ and, hence, the curve is $\mathbb{C}^\times/\zeta^{\mathbb{Z}}$. Moreover, we also have $2\chi_a = 0$ for every $a \in \mathbb{C}^\times$. In particular, $\widetilde{\Gamma}_\zeta$ is a quotient of $\widetilde{\Gamma}_1$ in this case. On the other hand, if $\mathfrak{g} = \mathfrak{sl}_3$ and $l = 3 = r^\vee h^\vee$, the curve remains the same as in the generic case.

We record the following lemma proved "in between the lines" in [12, 22] for ξ of infinite order. For $\xi \in \{\zeta, 1\}$ the proof is analogous. Given $i \in I$, let $\mathcal{P}_{\xi,i}$ be the subgroup of $\mathcal{P}_\xi$ generated by $\boldsymbol{\omega}_{i,a}$ with $a \in \mathbb{C}(\xi)^\times$ and $\mathcal{P}_{\xi,i}^+ = \mathcal{P}_\xi^+ \cap \mathcal{P}_{\xi,i}$. Similarly one defines $\mathcal{P}_{\xi,J}$ and $\mathcal{P}_{\xi,J}^+$ for any $J \subseteq I$.

Lemma 2.7. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{\xi,I_\bullet}^+$ are such that $\gamma_\xi(\boldsymbol{\lambda}) = \gamma_\xi(\boldsymbol{\mu})$, then there exists a sequence $\boldsymbol{\omega}_1, \ldots \boldsymbol{\omega}_m \in \mathcal{P}_{\xi,I_\bullet}^+$ such that $\boldsymbol{\omega}_1 = \boldsymbol{\lambda}, \boldsymbol{\omega}_m = \boldsymbol{\mu}$ and, for every $j = 1, \ldots, m-1$,

$$\frac{\boldsymbol{\omega}_j}{\boldsymbol{\omega}_{j+1}} = (\boldsymbol{\tau}_{\xi,k_j,a_j})^{\epsilon_j},$$

for some $k_j \in \{1,2,3\}, \epsilon_j \in \{\pm 1\}$, and $a_j \in \mathbb{C}(\xi)^\times$. Moreover, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{\mathbb{A}}^s$, then a_j can be taken in $\mathbb{A}^\times$ for all $j = 1, \ldots, m-1$. $\square$

3. The Irreducible and Highest-Weight Representations

3.1. **Weight Modules.** We now review the finite-dimensional representation theory of $U_\xi(\mathfrak{g})$. We include the sketch of some of the proofs which we believe to be relevant for the development of the paper. The omitted details on the material of this subsection can be found in [2, 15, 38, 40].

Let V be a $U_\xi(\mathfrak{g})$-module. Given $\mu \in P$ define

$$V_\mu = \left\{ v \in V : k_i v = \xi_i^{\mu(h_i)} v \text{ and } \left[\begin{smallmatrix} k_i \\ l \end{smallmatrix}\right] v = \left[\begin{smallmatrix} \mu(h_i) \\ l \end{smallmatrix}\right]_{\xi_i} v \right\}. \tag{3.1}$$

If ξ has infinite order, the second condition in the definition of V_μ is redundant. The space V_μ is said to be the weight space of V of weight μ and the nonzero vectors of μ are called weight vectors of weight μ. The module V is said to be a weight module (of type 1) if

$$V = \bigoplus_{\mu \in P} V_\mu.$$

It is well-known that every finite-dimensional $U_\xi(\mathfrak{g})$-module is isomorphic to the tensor product of a weight module of type 1 with a one-dimensional representation. We shall only consider representations of type 1 with finite-dimensional weight spaces and will refer to them simply as weight modules. The abelian tensor category of finite-dimensional $U_\xi(\mathfrak{g})$-weight modules will be denoted by $\mathcal{C}_\xi$.

Remark. Let V be a $U_\zeta(\mathfrak{g})$-weight module and note that k_i^l acts as the identity operator on V for all $i \in I$, and that $\zeta_i^{\mu(h_i)} \in \{\zeta_i^j : j = 0, 1, ..., l-1\}$ for every $i \in I$ and $\mu \in P$. Also, if $v \in V_\mu$ and $c \in \mathbb{Z}$, then $\left[\begin{smallmatrix} k_i;c \\ l \end{smallmatrix}\right] v = \left[\begin{smallmatrix} \mu(h_i)+c \\ l \end{smallmatrix}\right] v$. Furthermore, if $\mu(h_i) = m_0 + l m_1$ for some $m_0, m_1 \in \mathbb{Z}$, $0 \le m_0 \le l-1$, then $\zeta_i^{\mu(h_i)} = \zeta_i^{m_0}$ and $\left[\begin{smallmatrix} \mu(h_i) \\ l \end{smallmatrix}\right]_{\zeta_i} = m_1$ by Lemma 1.8.

Observe that the commutation relations (1.7) imply that

$$(x_i^{\pm})^{(k)} V_\mu \subseteq V_{\mu \pm k\alpha_i} \qquad \text{for all} \qquad k \in \mathbb{Z}_{\geq 0}, \quad \mu \in P. \tag{3.2}$$

Definition 3.1. A $U_\xi(\mathfrak{g})$-module V is said to be integrable if, given $v \in V$ and $i \in I$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(x_i^-)^{(k)} v = 0$ for all $k > m$.

Quite clearly, every finite-dimensional $U_\xi(\mathfrak{g})$-module is integrable.

If V is a weight module, define the character of V to be the element $\mathrm{char}(V)$ of the integral group ring $\mathbb{Z}[P]$ given by

$$\mathrm{char}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu. \tag{3.3}$$

Here, $e^\mu, \mu \in P$, denote the basis elements of $\mathbb{Z}[P]$ so that $e^\mu e^\nu = e^{\mu+\nu}$. The action of $\mathcal{W}$ on P can be naturally extended to an action of $\mathcal{W}$ on $\mathbb{Z}[P]$.

Proposition 3.2. Suppose V is an integrable $U_\xi(\mathfrak{g})$-module. Then, $\mathrm{char}(V)$ is $\mathcal{W}$-invariant.

Proof. Since V is integrable, the formula

$$\widetilde{T}_i v = \sum_{a,b,c,-a+b-c=\mu(h_i)} (-1)^b \xi_i^{-ac+b} (x_i^+)^{(a)} (x_i^-)^{(b)} (x_i^+)^{(c)} v$$

defines a linear operator on V. Quite clearly $\widetilde{T}_i(V_\mu) \subseteq V_{s_i\mu}$ for every $\mu \in P$. Proceeding as in [40, Chapter 5] one proves that $\widetilde{T}_i$ is a linear automorphism of V. $\square$

If $v \in V$ is a weight vector such that $U_\xi(\mathfrak{n}^+)^0 v = 0$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ, then V is said to be a highest-weight module of highest weight λ. The concept of lowest-weight module is defined similarly by replacing $U_\xi(\mathfrak{n}^+)^0$ with $U_\xi(\mathfrak{n}^-)^0$. If ξ is not a root of unity, the condition $U_\xi(\mathfrak{n}^+)^0 v = 0$ is equivalent to $x_i^+ v = 0$ for all $i \in I$. If $\xi = \zeta$, the condition $U_\xi(\mathfrak{n}^+)^0 v = 0$ is equivalent to $x_i^+ v = (x_i^+)^{(l)} v = 0$ for all $i \in I$ (cf. Lemma 1.9). The next proposition is standard.

Proposition 3.3. Let V be a highest-weight module. Then, V has a unique maximal proper submodule and, hence, a unique irreducible quotient. In particular, V is indecomposable. $\square$

Definition 3.4. Given $\lambda \in P^+$, let $W_\xi(\lambda)$ be the universal $U_\xi(\mathfrak{g})$-module with respect to the following property: $W_\xi(\lambda)$ is generated by a highest-weight vector v of weight λ satisfying $(x_i^-)^{(m)} v = 0$ for all $i \in I$ and $m > \lambda(h_i)$. The module $W_\xi(\lambda)$ is called the Weyl module of highest weight λ. The unique irreducible quotient of $W_\xi(\lambda)$ is denoted by $V_\xi(\lambda)$.

Remark. If ξ is not a root of unity, the condition $(x_i^-)^{(m)} v = 0$ for all $i \in I$ and $m > \lambda(h_i)$ is equivalent to $(x_i^-)^{\lambda(h_i)+1} v = 0$.

Theorem 3.5. For every $\lambda \in P^+$, $W_\xi(\lambda)$ is an integrable $U_\xi(\mathfrak{g})$-module. Furthermore, it is the universal finite-dimensional highest-weight $U_\xi(\mathfrak{g})$-module of highest weight λ.

Proof. Quite clearly the weight spaces of $W_\xi(\lambda)$ are finite-dimensional and integrability follows easily from the quantum Serre's relations and the condition $(x_i^-)^{(m)} v = 0$ for all $i \in I$ and $m > \lambda(h_i)$. Then, by Proposition 3.2, the set of weights of $W_\xi(\lambda)$ is invariant under the Weyl group action. In particular, $W_\zeta(\lambda)_\mu \neq 0$ only if $\mu \in P(\lambda)$ (cf. Lemma 1.1) which implies that $W_\zeta(\lambda)$ is finite-dimensional.

On the other hand, let V be a finite-dimensional highest-weight $U_\xi(\mathfrak{g})$-module of highest weight λ. Then, again, $V_\mu \neq 0$ only if $\mu \in P(\lambda)$ by Proposition 3.2. But this implies $V_{\lambda - 2k\alpha_i} = 0$ if $k > \lambda(h_i)$. Hence, V is a quotient of $W_\xi(\lambda)$. $\square$

It is quite easy to see that any finite-dimensional irreducible $U_\xi(\mathfrak{g})$-module (of type 1) is highest-weight. Moreover, since every element of P is conjugate to a dominant weight, it follows from Proposition 3.2 that the highest weight must be in P^+. This proves:

Corollary 3.6. Every simple object from $\mathcal{C}_\xi$ is isomorphic to $V_\xi(\lambda)$ for some $\lambda \in P^+$. □

The proof of the next theorem is standard.

Theorem 3.7. The irreducible constituents of $V \in \mathcal{C}_\xi$ (counted with multiplicities) are completely determined by $\mathrm{char}(V)$. □

The following result is now immediate from Lemma 1.1 and the definition of dual representation.

Corollary 3.8. For every $\lambda \in P^+$, $V_\xi(\lambda)$ is a lowest-weight module of lowest weight $w_0\lambda$. In particular, $V_\xi(\lambda)^* \cong V_\xi(-w_0\lambda)$. □

We also record the following theorem.

Theorem 3.9. For every $\lambda \in P^+$, $\mathrm{char}(W_\xi(\lambda))$ is given by the Weyl character formula. In particular, $W_\xi(\lambda)_\mu \neq 0$ iff $\mu \in P(\lambda)$. □

Remark. Due to Corollary 1.5, the above results for $U_1(\mathfrak{g})$ recover the basic results on the finite-dimensional representation theory of $\mathfrak{g}$. We shall use the notation $V(\lambda)$ for the $\mathfrak{g}$-module corresponding to $V_1(\lambda)$.

Theorem 3.10. Suppose ξ is not a root of unity and let V be an object from $\mathcal{C}_\xi$. Then, V is completely reducible. In particular, $V_\xi(\lambda) \cong W_\xi(\lambda)$ for every $\lambda \in P^+$. □

If $\xi = \zeta$, Theorem 3.10 is false and the category $\mathcal{C}_\zeta$ is not semisimple. For a brief overview of the character theory in this case, see [15, §11.2].

Let $P_l^+ = \{\lambda \in P^+ : \lambda(h_i) < l \text{ for all } i \in I\}$.

Proposition 3.11. If $\lambda \in P_l^+$, then $V_\zeta(\lambda)$ is irreducible as a $U_\zeta^{\rm fin}(\mathfrak{g})$-module. □

3.2. **ℓ-Weight Modules.** This section is based on [15, 19, 25] and we refer to these papers for the omitted details.

Let V be a $U_\xi(\tilde{\mathfrak{g}})$-module. A nonzero vector $v \in V$ is said to be an ℓ-weight vector if there exists $\boldsymbol{\lambda} \in \mathcal{P}_\xi$ and $k \in \mathbb{Z}_{>0}$ such that $(\eta - \boldsymbol{\Psi}_{\boldsymbol{\lambda}}(\eta))^k v = 0$ for all $\eta \in U_\xi(\tilde{\mathfrak{h}})$. In that case, $\boldsymbol{\lambda}$ is said to be the ℓ-weight of v. Let $V_{\boldsymbol{\lambda}}$ denote the subspace spanned by all ℓ-weight vectors of ℓ-weight $\boldsymbol{\lambda}$. V is said to be an ℓ-weight module if $V = \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_\xi} V_{\boldsymbol{\mu}}$. Denote by $\widetilde{\mathcal{C}}_\xi$ the category of all finite-dimensional $U_\xi(\tilde{\mathfrak{g}})$-$\ell$-weight modules. Quite clearly $\widetilde{\mathcal{C}}_\xi$ is an abelian category. Observe that if V is an ℓ-weight module, then V is also a $U_\xi(\mathfrak{g})$-weight module and

$$V_\lambda = \bigoplus_{\boldsymbol{\lambda} : \mathrm{wt}(\boldsymbol{\lambda}) = \lambda} V_{\boldsymbol{\lambda}}. \tag{3.4}$$

The following proposition was proved in [26, 25].

Proposition 3.12. Let $\xi \in \mathbb{C}' \backslash \{q\}$. If V is a finite-dimensional $U_\xi(\tilde{\mathfrak{g}})$-module which is a $U_\xi(\mathfrak{g})$-weight-module, then V is an ℓ-weight module. □

Remark. Note that not every finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module lying in $\mathcal{C}_q$ is an ℓ-weight module since $\mathbb{C}(q)$ is not algebraically closed. Instead, they are quasi-ℓ-weight modules in a sense similar to that developed in [32] for hyper loop algebras. We shall not need this concept here.

An ℓ-weight vector v of ℓ-weight $\boldsymbol\lambda$ is said to be a highest-ℓ-weight vector if $\eta v = \boldsymbol\Psi_{\boldsymbol\lambda}(\eta)v$ for every $\eta \in U_\xi(\tilde{\mathfrak h})$ and $U_\xi(\tilde{\mathfrak n}^+)^0 v = 0$. A module V is said to be a highest-ℓ-weight module if it is generated by a highest-ℓ-weight vector. The notion of a lowest-ℓ-weight module is defined similarly. If V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol\lambda$, then (1.7) implies

$$\dim(V_{\mathrm{wt}(\boldsymbol\lambda)}) = 1 \qquad \text{and} \qquad V_\mu \neq 0 \ \text{ only if } \ \mu \le \mathrm{wt}(\boldsymbol\lambda). \tag{3.5}$$

The next proposition is easily established using (3.5).

Proposition 3.13. If V is a highest-ℓ-weight module, then it has a unique proper submodule and, hence, a unique irreducible quotient. In particular, V is indecomposable. $\square$

Proposition 3.14. Let V be a finite-dimensional highest-ℓ-weight module of $U_\xi(\tilde{\mathfrak g})$ of highest ℓ-weight $\boldsymbol\lambda \in \mathcal P_\xi$, v a highest-ℓ-weight vector of V, and $\lambda = \mathrm{wt}(\boldsymbol\lambda)$. Then, $\boldsymbol\lambda \in \mathcal P_\xi^+$ and $(x_{i,r}^-)^{(m)}v = 0$ for all $i \in I, r \in \mathbb Z, m \in \mathbb Z_{>\lambda(h_i)}$. $\square$

Definition 3.15. Let $\boldsymbol\lambda \in \mathcal P_\xi^+$ and $\lambda = \mathrm{wt}(\boldsymbol\lambda)$. The Weyl module $W_\xi(\boldsymbol\lambda)$ of highest ℓ-weight $\boldsymbol\lambda$ is the universal $U_\xi(\tilde{\mathfrak g})$-module with respect to the following property: $W_\xi(\boldsymbol\lambda)$ is generated by a highest-ℓ-weight vector v of ℓ-weight $\boldsymbol\lambda$ satisfying $(x_{i,r}^-)^{(m)}v = 0$ for all $m > \lambda(h_i)$. Denote by $V_\xi(\boldsymbol\lambda)$ the irreducible quotient of $W_\xi(\boldsymbol\lambda)$.

Theorem 3.16. For every $\boldsymbol\lambda \in \mathcal P_\xi^+$, the module $W_\xi(\boldsymbol\lambda)$ is finite-dimensional. In particular, $W_\xi(\boldsymbol\lambda)$ is the universal finite-dimensional highest-ℓ-weight $U_\xi(\tilde{\mathfrak g})$-module of highest ℓ-weight $\boldsymbol\lambda$.

Proof. The latter statement is clear from the former and Proposition 3.14.

For $\xi = 1$ the theorem was proved in [20, §2]. For the remaining generic ξ it was proved in [20, §4] for simply laced $\mathfrak g$, while for $\mathfrak g$ with lacing number $r^\vee = 2$ it follows from the proof of [7, Proposition 2.2]. The proof for $\xi = \zeta$ and $r^\vee \le 2$ is similar. The case $\mathfrak g$ of type G_2 seems not to have been considered in the literature for $\xi \neq 1$ (however, see [5] for the connection with the so-called extremal weight modules). Below we review the main parts of the proof in general and provide the extra technical details for type G_2. In between the lines, we make minor corrections to the argument used in [7].

Set $V = W_\xi(\boldsymbol\lambda)$ and $\lambda = \mathrm{wt}(\boldsymbol\lambda)$ and let v be a highest-ℓ-weight vector of V. Proceeding as in the previous subsection, one easily proves that V is an integrable $U_\xi(\mathfrak g)$-module and, hence, has finitely many weight spaces. It remains to show that all weight spaces are finite-dimensional. The proof is similar to that of [20, Proposition 4.4]. Namely, we show by induction on the height of $\eta \in Q^+$ that there exists $N(\eta) \in \mathbb Z_{\ge 0}$ such that $V_{\lambda-\eta}$ is spanned by elements of the form $(x_{i_1,s_1}^-)^{(k_1)} \cdots (x_{i_m,s_m}^-)^{(k_m)}v$ with $\sum_j k_j\alpha_{i_j} = \eta$ and $|s_j| \le N(\eta)$ for all $j = 1, \ldots, m$. Clearly such vectors without the restriction on s_j span $V_{\lambda-\eta}$. Moreover, by Lemma 1.9, we can assume $k_j \in \{1, l\}$.

The proof of this in the case $\eta = k\alpha_i$ for some $i \in I$ and some $k \in \mathbb Z_{\ge 0}$ is similar to that given in [20, Proposition 4.4] or rather to that given in [31, Theorem 3.11] (in the context of hyper loop algebras) and we omit the details. In particular, this shows the induction on the height of η starts. We remark that the proof of this particular case makes use of Lemma 1.7. In [20, Proposition 4.4] only the special cases of the lemma mentioned in our proof above was used alongside multiplication by elements $h_{i,s}$ to shift the parameter s. In the root of unity case this multiplication has to be replaced by the more general version of Lemma 1.7 as stated here because $h_{i,s}$ may be zero (see Lemma 1.10).

Now suppose the height of η is larger that 1 and consider a vector of the form $(x_{i_1,s_1}^-)^{(k_1)} \cdots (x_{i_m,s_m}^-)^{(k_m)}v$ with $k_1 > 0$ and $\sum_{j>1} k_j > 0$. By the induction hypothesis we can assume that $|s_j| \le N(\eta - \alpha_{i_1})$ for $j > 1$. We will show that all such vectors are in the span of vectors as above with $|s_j| \le \max\{N(\eta - \alpha_i) : i \in I\} + r^\vee := N(\eta)$. If $i_1 = i_2$ this easily follows from [20, Lemma 4.3].

For $i_1 \neq i_2$ we perform a sub-induction on $N(\eta) - |s_1|$. Assume $\mathfrak g$ is of type G_2 and $c_{i_1,i_2} = -3$ (the other cases are similar and simpler). Fix the order $\{\alpha_{i_2}, \alpha_{i_1}+\alpha_{i_2}, 3\alpha_{i_1}+2\alpha_{i_2}, 2\alpha_{i_1}+\alpha_{i_2}, 3\alpha_{i_1}+\alpha_{i_2}, \alpha_{i_1}\}$

on R^+ and suppose that $s_1 \geq N(\eta)$. Then, by Theorem 1.16, (1.14), and Lemma 1.19, there exist $f, g : \mathbb{Z}^4 \to \mathbb{C}$ ($f, g : \mathbb{Z}^4 \to \mathbb{A}$ if $\xi = q$) such that

$$\begin{aligned}
(x^-_{i_1,s_1})^{(k_1)}(x^-_{i_2,s_2})^{(k_2)} &= \sum_{p,r,s,t\geq 0} f(p,r,s,t)(x^-_{i_2,s_2})^{(k_2-p-2r-s-t)}(\gamma^{s_1,s_2}_{\alpha_{i_1}+\alpha_{i_2}})^{(p)}(\gamma^{s_1,s_2}_{3\alpha_{i_1}+2\alpha_{i_2}})^{(r)}\times\\
&\quad\times(\gamma^{s_1,s_2}_{2\alpha_{i_1}+\alpha_{i_2}})^{(s)}(\gamma^{s_1,s_2}_{3\alpha_{i_1}+\alpha_{i_2}})^{(t)}(x^-_{i_1,s_1})^{(k_1-p-3r-2s-3t)}\\
&= \sum_{p,r,s,t\geq 0} g(p,r,s,t)(x^-_{i_2,s_2})^{(k_2-p-2r-s-t)}(\widetilde{\gamma}^{s_1-1,s_2+1}_{\alpha_{i_1}+\alpha_{i_2}})^{(p)}\times\\
&\quad\times([3]_\xi^{-1}[\widetilde{\gamma}^{s_1-1,s_2+1}_{\alpha_{i_1}+\alpha_{i_2}},\widetilde{\gamma}^{s_1-1,s_2+2}_{2\alpha_{i_1}+\alpha_{i_2}}]_\xi)^{(r)}\times\\
&\quad\times(\widetilde{\gamma}^{s_1-1,s_2+2}_{2\alpha_{i_1}+\alpha_{i_2}})^{(s)}(\widetilde{\gamma}^{s_1-1,s_2+3}_{3\alpha_{i_1}+\alpha_{i_2}})^{(t)}(x^-_{i_1,s_1})^{(k_1-p-3r-2s-3t)}
\end{aligned}\tag{3.6}$$

Notice that, since $l \neq 3$ and $k_1, k_2 \leq l$, we have $r < l$ above and, hence, $[3]_\xi^{-1}$ and $([r]_{\xi_i}!)^{-1}$ are indeed in $\mathbb{C}(\xi)$. This implies

$$([3]_\xi^{-1}[\widetilde{\gamma}^{s_1-1,s_2+1}_{\alpha_{i_1}+\alpha_{i_2}},\widetilde{\gamma}^{s_1-1,s_2+2}_{2\alpha_{i_1}+\alpha_{i_2}}]_\xi)^{(r)} = ([3]_\xi[r]_\xi!)^{-1}([\widetilde{\gamma}^{s_1-1,s_2+1}_{\alpha_{i_1}+\alpha_{i_2}},\widetilde{\gamma}^{s_1-1,s_2+2}_{2\alpha_{i_1}+\alpha_{i_2}}]_\xi)^r.$$

One now easily uses the induction hypotheses to complete the proof in this case. The case $s_1 \leq -N(\eta)$ is dealt with similarly. Namely, one first uses Corollary 1.16 to write

$$\begin{aligned}
(x^-_{i_1,s_1})^{(k_1)}(x^-_{i_2,s_2})^{(k_2)} &= \sum_{p,r,s,t\geq 0} f(p,r,s,t)(x^-_{i_2,s_2})^{(k_2-p-2r-s-t)}(\widetilde{\gamma}^{s_1,s_2}_{\alpha_{i_1}+\alpha_{i_2}})^{(p)}(\widetilde{\gamma}^{s_1,s_2}_{3\alpha_{i_1}+2\alpha_{i_2}})^{(r)}\times\\
&\quad\times(\widetilde{\gamma}^{s_1,s_2}_{2\alpha_{i_1}+\alpha_{i_2}})^{(s)}(\widetilde{\gamma}^{s_1,s_2}_{3\alpha_{i_1}+\alpha_{i_2}})^{(t)}(x^-_{i_1,s_1})^{(k_1-p-3r-2s-3t)}
\end{aligned}$$

for some function f as before and then uses (1.14) and Lemma 1.19 as before (this time Lemma 1.19 will cause s_1 to raise). □

Corollary 3.17. If $\xi \in \mathbb{C}'\backslash\{q\}$, every simple object $V \in \widetilde{\mathcal{C}}_\xi$ is isomorphic to $V_\xi(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathcal{P}^+$.

Proof. Since V is finite-dimensional and irreducible, it must be generated by one of its weight spaces corresponding to a maximal weight. Since $U_\xi(\tilde{\mathfrak{h}})$ is commutative, the irreducibility of V implies that this weight space is one-dimensional and, therefore, V is a highest ℓ-weight module. The corollary now follows immediately from Proposition 3.14 and Theorem 3.16. □

Remark. Note that not every irreducible finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module is highest ℓ-weight since $\mathbb{C}(q)$ is not algebraically closed. The classification of the irreducible finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules can be obtained using Galois theory similarly to what was done in [32] for the hyper loop algebras since the subalgebra of $U_{\mathbb{A}}(\tilde{\mathfrak{h}})$ generated by $\Lambda_{i,r}, r \in \mathbb{Z}_{\geq 0}$, is a polynomial algebra over $\mathbb{A}$ in these elements.

We shall need the following result about dual representations which was proved in [24, 25] for $\xi \neq 1$. For $\xi = 1$ it follows easily from Theorem 4.10 below.

Proposition 3.18. For every $\boldsymbol{\lambda} \in \mathcal{P}^+_\xi$, $V_\xi(\boldsymbol{\lambda})$ is a lowest-ℓ-weight module with lowest ℓ-weight $(\boldsymbol{\lambda}^*)^{-1}$. In particular, $V_\xi(\boldsymbol{\lambda})^* \cong V_\xi(\boldsymbol{\lambda}^*)$. □

Remark. Due to Corollary 1.5, the above results for $U_1(\tilde{\mathfrak{g}})$ recover the basic results on the finite-dimensional representation theory of $\tilde{\mathfrak{g}}$. We may use the notation $V(\boldsymbol{\lambda})$ and $W(\boldsymbol{\lambda})$ for the $\tilde{\mathfrak{g}}$-modules corresponding to $V_1(\boldsymbol{\lambda})$ and $W_1(\boldsymbol{\lambda})$, respectively.

We end this subsection recording the following analogue of Proposition 3.11.

Proposition 3.19. If $\boldsymbol{\lambda} \in \mathcal{P}^+_l$, then $V_\zeta(\boldsymbol{\lambda})$ is irreducible as a $U^{\text{fin}}_\zeta(\tilde{\mathfrak{g}})$-module. □

3.3. **The ℓ-characters.** Let $\mathbb{Z}[\mathcal{P}_\xi]$ be the integral group ring over $\mathcal{P}_\xi$. If V is an ℓ-weight module, the ℓ-character of V is defined to be the following element of $\mathbb{Z}[\mathcal{P}_\xi]$:

$$\mathrm{char}_\ell(V) = \sum_{\boldsymbol{\mu}\in\mathcal{P}_\xi} \dim(V_{\boldsymbol{\mu}})\boldsymbol{\mu}. \tag{3.7}$$

The valuation map ϵ_ξ induces a ring homomorphism $\mathbb{Z}[\mathcal{P}_q] \to \mathbb{Z}[\mathcal{P}]$ also denoted by ϵ_ξ.

Remark. The ℓ-characters were originally defined in [26, 25] where they were called q-characters and ζ-characters, respectively. To keep a more uniform terminology, we prefer to call them ℓ-characters as in [12] which stresses the fact that they are generating functions for the dimensions of the ℓ-weight spaces.

We consider the following partial order on $\mathbb{Z}[\mathcal{P}_\xi]$. Given $\eta\in\mathbb{Z}[\mathcal{P}_\xi]$, write $\eta = \sum_{\boldsymbol{\mu}\in\mathcal{P}_\xi}\eta(\boldsymbol{\mu})\boldsymbol{\mu}$. Then,

$$\eta\le\eta' \qquad \text{iff} \qquad \eta(\boldsymbol{\mu})\le\eta'(\boldsymbol{\mu}) \qquad \text{for all} \qquad \boldsymbol{\mu}\in\mathcal{P}_\xi.$$

We shall say η is an upper (respectively lower) bound for $\mathrm{char}_\ell(V)$ provided $\mathrm{char}_\ell(V)\le\eta$ (respectively $\eta\le\mathrm{char}_\ell(V)$).

The proof of the following theorem is analogous to that of Theorem 3.7.

Theorem 3.20. Let $V\in\widetilde{\mathcal{C}}_\xi$. The irreducible constituents (counted with multiplicities) of V are completely determined by $\mathrm{char}_\ell(V)$. □

It turns out that the ℓ-characters of objects from $\widetilde{\mathcal{C}}_\xi$ are not invariant with respect to the braid group action on $\mathcal{P}_\xi$ in general. In fact, the theory of ℓ-characters is much more intricate than that of characters and it is not yet completely understood unless $\xi=1$. Even the following basic theorem proved in [24, 25] (for $\xi\ne1$) is not simple to be proved (see also [12] for a different approach when $\mathfrak{g}$ is of classical type). For $\xi=1$ this theorem follows easily from Theorem 4.10 below (see also [11]).

Theorem 3.21. Let V be a finite-dimensional highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\lambda}\in\mathcal{P}_\xi^+$. If $\boldsymbol{\lambda}_i(u)$ splits in $\mathbb{C}(\xi)[u]$ for every $i\in I$, then V is an ℓ-weight module and $V_{\boldsymbol{\mu}}\ne0$ only if $\boldsymbol{\mu}\le\boldsymbol{\lambda}$. More precisely, if $\boldsymbol{\lambda}=\prod_{j=1}^m\boldsymbol{\omega}_{\lambda_j,a_j}$ for some $\lambda_j\in P^+$ and $a_j\in\mathbb{C}(\xi)^\times$ and if $V_{\boldsymbol{\mu}}\ne0$, then $\boldsymbol{\mu}=\boldsymbol{\lambda}\left(\prod_{k=1}^{m'}\boldsymbol{\alpha}_{i_k,a_{j_k}\xi^{b_k}}\right)^{-1}$ for some $i_k\in I, j_k\in\{1,\dots,m\}$, and $0\le b_k<r^\vee h^\vee$. In particular, $\boldsymbol{\mu}=\prod_{k=1}^{m''}\boldsymbol{\omega}_{i'_k,a_{j'_k}\xi^{b'_k}}$ for some $i'_k\in I, j'_k\in\{1,\dots,m\}$, and $0\le b'_k\le r^\vee h^\vee$. □

Remark. If $\boldsymbol{\lambda}_i(u)$ does not split for some $i\in I$ (in particular $\xi=q$), V is not an ℓ-weight module, but rather a quasi-ℓ-weight module, as mentioned previously. An analogue of the above theorem in this case can be established by following the ideas of [32]. Observe that it follows that $W_q(\boldsymbol{\lambda})\in\widetilde{\mathcal{C}}_q$ iff the roots of $\boldsymbol{\lambda}_i(u)$ (if any) are in $\mathbb{C}(q)$ for all $i\in I$.

One easily sees from Proposition 1.6(d) that

$$\mathrm{char}_\ell(V\otimes W) = \mathrm{char}_\ell(V)\mathrm{char}_\ell(W) \tag{3.8}$$

for every $V,W\in\widetilde{\mathcal{C}}_\xi$. In particular, this and Theorem 3.20 imply:

Corollary 3.22. Let $V_1,\dots,V_m\in\widetilde{\mathcal{C}}_\xi$ and $\sigma\in S_m$. The irreducible constituents of $V_{\sigma(1)}\otimes\cdots\otimes V_{\sigma(m)}$ (counted with multiplicities) do not depend on σ. □

Remark. While $\mathcal{C}_\xi$ is a braided tensor category for any value of ξ, $\widetilde{\mathcal{C}}_\xi$ is not braided when $\xi\ne1$.

4. Specializations, Highest-ℓ-Weight Tensor Products, and Applications

4.1. **Specialization of modules.** In this subsection, let $\xi \in \mathbb{C}'\backslash\{q\}$. Recall that an $\mathbb{A}$-lattice of a $\mathbb{C}(q)$-vector space V is the $\mathbb{A}$-span of a basis of V. If V is a $U_q(\tilde{\mathfrak{g}})$-module (resp. $U_q(\mathfrak{g})$-module) and L is an $\mathbb{A}$-lattice of V, we will say that L is a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice (resp. $U_{\mathbb{A}}(\mathfrak{g})$-admissible lattice) if it is a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-submodule (resp. $U_{\mathbb{A}}(\mathfrak{g})$-submodule) of V. Given a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice (resp. $U_{\mathbb{A}}(\mathfrak{g})$-admissible lattice) of a module V, define

$$\bar{L} = L \otimes_{\mathbb{A}} \mathbb{C}_\xi. \tag{4.1}$$

Then $\bar{L}$ is a $U_\xi(\tilde{\mathfrak{g}})$-module (resp. $U_\xi(\mathfrak{g})$-module) and $\dim_{\mathbb{C}}(\bar{L}) = \dim_{\mathbb{C}(q)}(V)$. Given $v \in L$, we shall denote an element of the form $v \otimes 1 \in \bar{L}$ simply by v. The proof of the following proposition can be found in [36, Proposition 4.2].

Proposition 4.1.

(a) Let $V \in \mathcal{C}_q$ and L be a $U_{\mathbb{A}}(\mathfrak{h})$-submodule of V. Then, $L = \bigoplus\limits_{\mu \in P} L \cap V_\mu$. In particular, if L is a $U_{\mathbb{A}}(\mathfrak{g})$-admissible lattice of V, then $\operatorname{char}(\bar{L}) = \operatorname{char}(V)$.

(b) Let $\lambda \in P^+$, v a highest-weight vector of $V_q(\lambda)$, and $L = U_{\mathbb{A}}(\mathfrak{g})v$. Then, L is a $U_{\mathbb{A}}(\mathfrak{g})$-admissible lattice of V. □

Corollary 4.2. Let L be as in Proposition 4.1(b). Then, $\bar{L} \cong W_\xi(\lambda)$.

Proof. Quite clearly $\bar{L}$ is a highest-weight $U_\xi(\mathfrak{g})$-module and, hence, a quotient of $W_\xi(\lambda)$. The isomorphism now follows from Proposition 4.1 and Theorems 3.9 and 3.10. □

Recall the definitions of $\mathcal{P}_{\mathbb{A}}^{++}$ and $\mathcal{P}_{\mathbb{A}}^{s}$ given in §2.1. Recall also that, for $\boldsymbol{\mu} \in \mathcal{P}_{\mathbb{A}}$, $\overline{\boldsymbol{\mu}}$ denotes the element of $\mathcal{P}$ obtained from $\boldsymbol{\mu}$ by applying the evaluation map ϵ_ξ.

Theorem 4.3. Let V be a $U_q(\tilde{\mathfrak{g}})$-highest-$\ell$-weight module with highest-ℓ-weight $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbb{A}}^{++}$, v a highest-ℓ-weight vector of V, and $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then, L is a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of $V, \bar{L}$ is a quotient of $W_\xi(\overline{\boldsymbol{\lambda}})$, and $\operatorname{char}(\bar{L}) = \operatorname{char}(V)$.

Proof. This is essentially a corollary of the proof of Theorem 3.16 (with $\xi = q$) as observed in [20]. However, the case $\mathfrak{g}$ of type G_2 is reasonably more subtle and it seems that it has not been considered before. Namely, L is clearly invariant under the action of $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ and we shall see that it follows from the proof of Theorem 3.16 that L is finitely generated over $\mathbb{A}$. Assuming this, since $\mathbb{A}$ is a principal ideal domain and L is obviously torsion free, it follows that L is a free $\mathbb{A}$-module. Moreover, since $\mathbb{C}(q)$ is the fraction field of $\mathbb{A}$, an $\mathbb{A}$-basis of L is linearly independent over $\mathbb{C}(q)$ showing that the rank of L is less than or equal to the dimension of V. On the other hand, it follows from the proof of Theorem 3.16 again that L contains a $\mathbb{C}(q)$-basis of V. The first statement follows immediately, while the second follows after observing that $\bar{L}$ is a highest-ℓ-weight module of highest ℓ-weight $\overline{\boldsymbol{\lambda}}$. The third statement is immediate from the first together with Proposition 4.1(a).

If $\mathfrak{g}$ is not of type G_2, proceeding as in the proof of Theorem 3.16 it immediately follows that L is generated over $\mathbb{A}$ by finitely many elements of the form $(x_{i_1,r_1}^-)^{(k_1)} \cdots (x_{i_m,r_m}^-)^{(k_m)} v$ with $(i_j, s_j, k_j) \in I \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. If $\mathfrak{g}$ is of type G_2, let us take a closer look at (3.6) (and its analogue for $s_1 < 0$ which we shall not write down):

$$\begin{aligned}(x_{i_1,s_1}^-)^{(k_1)}(x_{i_2,s_2}^-)^{(k_2)} = \sum_{p,r,s,t\geq 0} & g(p,r,s,t)(x_{i_2,s_2}^-)^{(k_2-p-2r-s-t)}(\tilde{\gamma}_{\alpha_{i_1}+\alpha_{i_2}}^{s_1-1,s_2+1})^{(p)}\times \\ & \times([3]_q^{-1}[\tilde{\gamma}_{\alpha_{i_1}+\alpha_{i_2}}^{s_1-1,s_2+1},\tilde{\gamma}_{2\alpha_{i_1}+\alpha_{i_2}}^{s_1-1,s_2+2}]_q)^{(r)}\times \\ & \times(\tilde{\gamma}_{2\alpha_{i_1}+\alpha_{i_2}}^{s_1-1,s_2+2})^{(s)}(\tilde{\gamma}_{3\alpha_{i_1}+\alpha_{i_2}}^{s_1-1,s_2+3})^{(t)}(x_{i_1,s_1}^-)^{(k_1-p-3r-2s-3t)}\end{aligned} \tag{3.6}$$

Applying Lemma 1.19 we also get

$$(x^-_{i_1,s_1})^{(k_1)}(x^-_{i_2,s_2})^{(k_2)} = \sum_{p,r,s,t\geq 0} g'(p,r,s,t)(x^-_{i_2,s_2})^{(k_2-p-2r-s-t)}(\tilde\gamma^{s_1-1,s_2+1}_{\alpha_{i_1}+\alpha_{i_2}})^{(p)}\times$$

$$\times\, (\gamma^{s_1,s_2}_{3\alpha_{i_1}+2\alpha_{i_2}})^{(r)}(\tilde\gamma^{s_1-1,s_2+2}_{2\alpha_{i_1}+\alpha_{i_2}})^{(s)}(\tilde\gamma^{s_1-1,s_2+3}_{3\alpha_{i_1}+\alpha_{i_2}})^{(t)}(x^-_{i_1,s_1})^{(k_1-p-3r-2s-3t)} \tag{4.2}$$

for some $g' : \mathbb{Z}^4 \to \mathbb{A}$. Observe that if (k_1, k_2) is not of the form $(3k, 2k)$, then $r < k_1/3$ in (4.2) and the induction hypotheses of the proof of Theorem 3.16 apply and imply that the element $(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v$ belongs to the $\mathbb{A}$-span of similar elements with $|s_j| \leq N(\eta)$ where $\eta = \sum_j k_j\alpha_{i_j}$. Thus, assume $(k_1, k_2) = (3k, 2k)$. By multiplying (3.6) by $([3]_q)^k[k]_q!$ and using the induction hypotheses on the height of η and on s_1 as before, one easily sees that the elements $([3]_q)^k[k]_q!(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v$ belong to the $\mathbb{A}$-span of elements of the form $(x^-_{i'_1,s'_1})^{(k'_1)}\cdots(x^-_{i'_m,s'_m})^{(k'_m)}v$ with $s'_j < N(\eta)$.

Let L' be the $\mathbb{A}$-span of the elements $(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v$ with $(k_1, k_2) \neq (3k, 2k)$ and of the elements $([3]_q)^k[k]_q!(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v$ with $(k_1, k_2) = (3k, 2k)$ for some $k \geq 1$. It follows that L' is contained in the $\mathbb{A}$-span of elements of the form $(x^-_{i'_1,s'_1})^{(k'_1)}\cdots(x^-_{i'_m,s'_m})^{(k'_m)}v$ with $s'_j < N(\eta)$. Hence, L' is a finitely generated $\mathbb{A}$-submodule of V. On the other hand, it is clear that L' spans V over $\mathbb{C}(q)$ and, hence, we conclude that L' is an $\mathbb{A}$-lattice of V as before.

Let $H = \max\{\text{height}(\eta) : \eta \in Q^+, V_{\text{wt}(\boldsymbol\lambda)-\eta} \neq 0\}$. Clearly $(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v \neq 0$ only if $k_1 \leq H$. Given $k, s \in \mathbb{Z}_{>0}, k \leq H$, let $L_{k,s}$ be the $\mathbb{A}$-module generated by L' and the elements $(x^-_{i_1,s_1})^{(k_1)}\cdots(x^-_{i_m,s_m})^{(k_m)}v$ with $(k_1, k_2) = (3k', 2k')$ for some $k' \leq k$, $s_1 = s'$ for some $s' \leq s$ and such $\eta = \sum_j k_j\alpha_{i_j}$ satisfies $V_{\text{wt}(\boldsymbol\lambda)-\eta} \neq 0$. Clearly $L = \cup_{k,s}L_{k,s}$ and $L' \subseteq L$. On the other hand, $(([3]_q)^k[k]_q!)L_{k,s} \subseteq L'$ by (3.6) and, hence, $([3]_q)^H[H]_q!L \subseteq L'$. Since L' is free of finite rank, it follows that $(([3]_q)^H[H]_q!)L$ is free of finite rank. Finally, $(([3]_q)^H[H]_q!)L$ is clearly isomorphic to L as an $\mathbb{A}$-module and it follows that L is a finitely generated $\mathbb{A}$-module containing a basis of V. ☐

From now on, given a highest-ℓ-weight module V as in the above proposition, we will denote by $\overline{V}$ the $U_\xi(\tilde{\mathfrak{g}})$-module $\bar L$ as constructed there.

Proposition 4.4. Let L be a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of $V \in \widetilde{\mathcal{C}}_q$. Then, $\text{char}_\ell(\overline{L}) = \epsilon_\xi(\text{char}_\ell(V))$.

Proof. Let $B = \{v_1, \ldots, v_m\}$ be an $\mathbb{A}$-basis of L and, hence, also a $\mathbb{C}(q)$-basis of V. Given $i \in I$ and $r \in \mathbb{Z}$, let $A_{i,r}$ be the matrix of the action of $\Lambda_{i,r}$ on V with respect to B. Then, the entries $a_{s,t}$ of $A_{i,r}$ are in $\mathbb{A}$ and the characteristic polynomial $p(u)$ of the action of $\Lambda_{i,r}$ on V is $p(u) = \det(u\text{Id} - A_{i,r}) \in \mathbb{A}[u]$. It follows that the entries of the matrix of the action of $\Lambda_{i,r}$ on $\overline{L}$ with respect to the basis $\overline{B}$ are $\epsilon_\xi(a_{s,t})$ and, hence, the characteristic polynomial is $\overline{p(u)}$. The proposition follows. ☐

Remark. A version of Proposition 4.4 was stated in [25, Theorem 3.2] for V irreducible and L as in Theorem 4.3. However, the statement of [25, Theorem 3.2] is missing the hypothesis $\boldsymbol\lambda \in \mathcal{P}^s_{\mathbb{A}}$. This hypothesis is necessary since otherwise $V \notin \widetilde{\mathcal{C}}_q$ and hence $V \supsetneqq \bigoplus_{\boldsymbol\mu} V_{\boldsymbol\mu}$. In Example 4.34 below we shall see that the "loop" analogue of the first statement of Proposition 4.1(a) does not hold. Namely, if L is an $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-admissible lattice of $V \in \widetilde{\mathcal{C}}_q$, it may not be true that $L = \bigoplus_{\boldsymbol\mu\in\mathcal{P}_q} L \cap V_{\boldsymbol\mu}$ even if V is a highest-ℓ-weight module with highest ℓ-weight $\boldsymbol\lambda \in \mathcal{P}^s_{\mathbb{A}}$ and L is as in Theorem 4.3.

Given $V \in \widetilde{\mathcal{C}}_\xi$, set

$$\text{wt}_\ell(V) = \{\boldsymbol\mu \in \mathcal{P}_\xi : V_{\boldsymbol\mu} \neq 0\}. \tag{4.3}$$

Corollary 4.5. Let $\boldsymbol\lambda \in \mathcal{P}^{++}_{\mathbb{A}}$ and suppose $\epsilon_\xi(\text{wt}_\ell(V_q(\boldsymbol\lambda))) \cap \mathcal{P}^+_\xi = \{\overline{\boldsymbol\lambda}\}$. Then, $\overline{V_q(\boldsymbol\lambda)} \cong V_\xi(\overline{\boldsymbol\lambda})$.

Proof. Since $V_\xi(\overline{\boldsymbol\lambda})$ is a quotient of $\overline{V_q(\boldsymbol\lambda)}$, it suffices to show that the latter is irreducible. If this were not the case, $\overline{V_q(\boldsymbol\lambda)}$ would have to contain an irreducible submodule and, hence, $\epsilon_\xi(\mathrm{wt}_\ell(V_q(\boldsymbol\lambda)))$ would have to contain an element of $\mathcal{P}_\xi^+$ other than $\overline{\boldsymbol\lambda}$. □

Remark. Notice that Theorem 3.21 implies that, if $\mathrm{wt}_\ell(V_q(\boldsymbol\lambda))\cap\mathcal{P}_q^+=\{\boldsymbol\lambda\}$ and either ξ is not a root of unity or $\xi=\zeta$ and $l>2r^\vee h^\vee$, then the hypothesis of the above corollary is satisfied.

Let $\mathfrak g=\mathfrak{sl}_2$, $I=\{i\}$, v be a highest-weight vector of $V_q(\lambda),\lambda\in P^+$, and $L=U_{\mathbb A}(\mathfrak g)v$. Then it is easy to see that

$$\{(x_i^-)^{(k)}v : 0\le k\le\lambda(h_i)\}\quad\text{is an } \mathbb A\text{-basis of } L \text{ and}\quad (x_i^+)^{(\lambda(h_i))}(x_i^-)^{(\lambda(h_i))}v=v. \tag{4.4}$$

In particular, the image of this set in $\bar L$ is a $\mathbb C$-basis of $\bar L$. Now let $\mathfrak g$ be general again, v and L be as in Theorem 4.3, and $\lambda=\mathrm{wt}(\boldsymbol\lambda)$. Fix a reduced expression $s_{i_N}\cdots s_{i_1}$ for $w\in\mathcal W$, and consider

$$m_j=(s_{i_{j-1}}\cdots s_{i_1}\lambda)(h_{i_j}). \tag{4.5}$$

Proceeding inductively on j, it follows from Lemma 1.1(b) and (4.4) that $m_j\in\mathbb Z_{\ge0}$,

$$(x_{i_1}^+)^{(m_1)}\cdots(x_{i_j}^+)^{(m_j)}(x_{i_j}^-)^{(m_j)}\cdots(x_{i_1}^-)^{(m_1)}v=v, \tag{4.6}$$

and $\{(x_{i_j}^-)^{(m_j)}\cdots(x_{i_1}^-)^{(m_1)}v\}$ is an $\mathbb A$-basis of $L\cap V_{s_{i_j}\cdots s_{i_1}\lambda}$. Set

$$v_w=(x_{i_N}^-)^{(m_N)}\cdots(x_{i_1}^-)^{(m_1)}v. \tag{4.7}$$

Proposition 4.6. Let $V,v,w,\lambda,\boldsymbol\lambda,m_j$ be as above. Then, $V_{T_w\boldsymbol\lambda}=V_{w\lambda}=\mathbb Cv_w$.

Proof. It remains to show that $v_w\in V_{T_w\boldsymbol\lambda}$. This was done in [8, Proposition 4.1]. □

Corollary 4.7. Let V be a finite-dimensional highest-ℓ-weight $U_\xi(\tilde{\mathfrak g})$-module of highest ℓ-weight $\boldsymbol\lambda\in\mathcal P_\xi^+$. Then, $V_{T_w\boldsymbol\lambda}=V_{w\mathrm{wt}(\boldsymbol\lambda)}$ for all $w\in\mathcal W$.

Proof. For $\xi=q$ this is Proposition 4.6 again. Otherwise, since $\mathcal P_\xi^+\subseteq\mathcal P_{\mathbb A}^s$, the result is immediate from Propositions 4.6 and 4.4. □

Corollary 4.8. Let $\xi\in\mathbb C'$ and $\boldsymbol\lambda\in\mathcal P_\xi^+$ be such that $\boldsymbol\lambda_i(u)$ splits in $\mathbb C(q)[u]$ for all $i\in I$ and suppose that $i_0\in I$ and $w\in\mathcal W$ are such that $\ell(s_{i_0}w)=\ell(w)+1$. Then, $T_w\boldsymbol\lambda\le\boldsymbol\lambda$ and $(T_w\boldsymbol\lambda)_{i_0}(u)\in\mathbb C(\xi)[u]$ splits. Moreover, the roots of $T_w\boldsymbol\lambda_{i_0}(u)$ form a subset of $\cup_k a_k\xi^{\mathbb Z}$ where a_k runs through the set of roots of $\boldsymbol\lambda_i(u),i\in I$.

Proof. The proof goes inductively on $\ell(w)$. If $\ell(w)=0$ there is nothing to be done. Otherwise, let $j\in I$ and $w'\in\mathcal W$ be such that $w=s_jw'$ and $\ell(w')=\ell(w)-1$. By the induction hypothesis, $T_{w'}\boldsymbol\lambda\le\boldsymbol\lambda$. In particular, by the very definition of $\boldsymbol\alpha_{i,a}$, $T_{w'}\boldsymbol\lambda=\boldsymbol\mu\boldsymbol\nu^{-1}$ for unique relatively prime $\boldsymbol\mu,\boldsymbol\nu\in\mathcal P_\xi^+$ such that $\boldsymbol\mu_i(u),\boldsymbol\nu_i(u)$ split in $\mathbb C(\xi)[u]$ for all $i\in I$. Also by the induction hypothesis, $(T_{w'}\boldsymbol\lambda)_j(u)\in\mathbb C(\xi)[u]$, i.e., $\boldsymbol\nu_j(u)=1$ and the roots of $\boldsymbol\mu_j(u)$ lie in the set $\cup_k a_k\xi^{\mathbb Z}$.

Let $V=V_\xi(\boldsymbol\lambda)$ and v_w be defined as in (4.7). It follows from Lemma 1.1(b) that $U_\xi(\tilde{\mathfrak n}_j^+)^0v_{w'}=0$ and, hence, that $U_\xi(\tilde{\mathfrak g}_j)v_{w'}$ is finite-dimensional $U_\xi(\tilde{\mathfrak g}_j)$-highest-$\ell$-weight module of highest-ℓ-weight $(T_{w'}\boldsymbol\lambda)_j(u)$. It now follows from Theorem 3.21 that $T_w\boldsymbol\lambda\le T_{w'}\boldsymbol\lambda$. Moreover, by Lemma 1.1(b) again, $U_\xi(\tilde{\mathfrak n}_{i_0}^+)^0v_w=0$ and $U_\xi(\tilde{\mathfrak g}_{i_0})v_w$ is a finite-dimensional $U_\xi(\tilde{\mathfrak g}_{i_0})$-highest-$\ell$-weight module of highest-ℓ-weight $(T_w\boldsymbol\lambda_{i_0})(u)$. In particular, $(T_w\boldsymbol\lambda)_{i_0}(u)$ is polynomial which splits in $\mathbb C(\xi)[u]$. Using Theorem 3.21 once more the last statement follows. □

Remark. For ξ of infinite order, the above corollary was first stated in [12]. As observed in [9], the proof of [12] was incomplete. The approach we used above is a representation theoretic alternative to the one presented in [9, §2.10] which relies instead purely on the Weyl group action on P and the braid group action on $\mathcal{P}_\xi$.

Conjecture 4.9. Let $\boldsymbol{\lambda} \in \mathcal{P}^+$ and $V = W_q(\boldsymbol{\lambda})$. Then, $\overline{V} \cong W_\xi(\boldsymbol{\lambda})$.

Conjecture 4.9 is the natural affine analogue of Corollary 4.2. In the case $\xi = 1$, it was originally conjectured in [20] and proved in [10] for $\mathfrak{g}$ of type A and in [23] for simply laced $\mathfrak{g}$. Moreover, it was pointed out by Nakajima that the results of [34, 35, 41] can be used to prove the conjecture in the case $\xi = 1$. We shall leave the discussion about the general case of the above conjecture to the future. We only mention that in the case $\xi = 1$ the proof can be reduced to the case $\boldsymbol{\lambda} = \boldsymbol{\omega}_{\lambda,a}$ with $\lambda \in P^+$ and $a \in \mathbb{C}^\times$. This follows from the second statement of part (b) of Theorem 4.10 below. If $\xi = \zeta$, however, it is unclear to us at this moment if such a reduction is possible.

4.2. **Evaluation modules.** Given a $\mathfrak{g}$-module V, let $V(a)$ be the $\tilde{\mathfrak{g}}$-module obtained by pulling-back the evaluation map $\mathrm{ev}_a : \tilde{\mathfrak{g}} \to \mathfrak{g}, a \in \mathbb{C}^\times$. Such modules are called evaluation modules. If $V = V(\lambda)$ we use the notation $V(\lambda, a)$ for the corresponding evaluation module. The next theorem can be found in [20].

Theorem 4.10. Let $\boldsymbol{\lambda} \in \mathcal{P}^+$.

(a) If $\boldsymbol{\lambda} = \boldsymbol{\omega}_{\lambda,a}$ for some $\lambda \in P^+$ and some $a \in \mathbb{C}^\times$, then $V(\boldsymbol{\lambda}) \cong V(\lambda, a)$.
(b) If $\boldsymbol{\lambda} = \prod_j \boldsymbol{\omega}_{\lambda_j,a_j}$ with $\lambda_j \in P^+$ and $a_i \neq a_j$ when $i \neq j$, then $V(\boldsymbol{\lambda}) \cong \bigotimes_j V(\lambda_j, a_j)$ and $W(\boldsymbol{\lambda}) \cong \bigotimes_j W(\boldsymbol{\omega}_{\lambda_j,a_j})$. □

In the remainder of this subsection we assume $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\xi \in \mathbb{C}' \backslash \{q\}$. Consider the $\mathbb{C}(q)$-algebra $U_q'(\mathfrak{g})$ given by generators $x_i^\pm, k_\mu^{\pm 1}$ with $i \in I, \mu \in P$, and the following defining relations:

$$k_\mu k_\mu^{-1} = k_\mu^{-1} k_\mu = 1, \qquad k_\mu k_\nu = k_{\mu+\nu}$$

$$k_\mu x_j^\pm k_\mu^{-1} = q^{\pm\mu(h_j)} x_j^\pm, \qquad [x_i^+, x_j^-] = \delta_{i,j} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k (x_i^\pm)^{(1-c_{ij}-k)} x_j^\pm (x_i^\pm)^{(k)} = 0, \quad \text{if } i \neq j,$$

There is an obvious monomorphism of algebras $U_q(\mathfrak{g}) \to U_q'(\mathfrak{g})$ such that $k_i \mapsto k_{\alpha_i}$. A representation V of $U_q'(\mathfrak{g})$ is said to be of type 1 if the generators $k_\nu, \nu \in P$, act diagonally with eigenvalues of the form $q^{(\nu,\mu)}$ for some $\mu \in P$ where $(\cdot,\cdot)$ is the bilinear form such that $(\alpha_i, \alpha_j) = c_{ij}$. It is not difficult to see that restriction establishes an equivalence of categories from that of type 1 finite-dimensional $U_q'(\mathfrak{g})$-modules to $\mathcal{C}_q$. From now on we shall identify these two categories using this equivalence. One can construct the algebras $U_{\mathbb{A}}'(\mathfrak{g})$ and $U_\xi'(\mathfrak{g})$ similarly as $U_{\mathbb{A}}(\mathfrak{g})$ and $U_\xi(\mathfrak{g})$. The next proposition was proved in [33, §2] and [16, Proposition 3.4].

Theorem 4.11. Suppose $\mathfrak{g}$ is of type A_n. There exists an algebra homomorphism $\mathrm{ev} : U_q(\tilde{\mathfrak{g}}) \to U_q'(\mathfrak{g})$ such that, if $\lambda \in P^+$ and V is the pull-back of $V_q(\lambda)$ by ev, then there exists $m(\lambda) \in \mathbb{Z}$ such that V is isomorphic to $V_q(\boldsymbol{\lambda})$ where

$$\boldsymbol{\lambda} = \prod_{i \in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \qquad \text{with} \qquad a_1 = q^{m(\lambda)} \quad \text{and} \quad \frac{a_{i+1}}{a_i} = q^{\lambda(h_i)+\lambda(h_{i+1})+1} \quad \text{for} \quad i < n.$$

Moreover, the image of $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ by ev is contained in $U_{\mathbb{A}}'(\mathfrak{g})$. □

Given $a \in \mathbb{C}(q)^\times$, let

$$\mathrm{ev}_a = \mathrm{ev} \circ \varrho_a \tag{4.8}$$

where ϱ_a is defined in Proposition 1.2. Denote by $V_q(\lambda, a)$ the pull-back of $V_q(\lambda)$ by the evaluation map ev_a. It is easy to see that $V_q(\lambda, a) \cong V_q(\boldsymbol{\lambda})$ where

$$\boldsymbol{\lambda} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \qquad \text{with} \qquad a_1 = aq^{m(\lambda)} \quad \text{and} \quad \frac{a_{i+1}}{a_i} = q^{\lambda(h_i)+\lambda(h_{i+1})+1} \quad \text{for} \quad i < n. \tag{4.9}$$

It turns out that, for $\mathfrak{g}$ not of type A, there is no analogue of Theorem 4.11. In fact, it is known (see [7] for instance) that there exists $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$ such that the action of $U_q(\mathfrak{g})$ on $V_q(m\omega_i)$ cannot be extended to one of $U_q(\tilde{\mathfrak{g}})$.

Given $a \in \mathbb{C}^\times$ and $\lambda \in P^+$, let $\mu = -w_0\lambda$ and $b = aq^{-n-1} = aq^{-r^\vee h^\vee}$. Then, it is not difficult to see that $V_q(\mu, b)^*$ is isomorphic to $V_q(\boldsymbol{\lambda})$ where $\boldsymbol{\lambda}$ is such that

$$\boldsymbol{\lambda} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \qquad \text{with} \qquad a_n = aq^{m(\mu)} \quad \text{and} \quad \frac{a_{i+1}}{a_i} = q^{-(\lambda(h_i)+\lambda(h_{i+1})+1)} \quad \text{for} \quad i < n. \tag{4.10}$$

Denote also by ev the induced map $\mathrm{ev} : U_\xi(\tilde{\mathfrak{g}}) \to U'_\xi(\mathfrak{g})$. Similarly, given $a \in \mathbb{C}^\times$, one defines the evaluation map $\mathrm{ev}_a : U_\xi(\tilde{\mathfrak{g}}) \to U'_\xi(\mathfrak{g})$. Denote by $V_\xi(\lambda, a)$ the pull-back of $V_\xi(\lambda)$ by ev_a and, similarly, let $W_\xi(\lambda, a)$ be the pull-back of $W_\xi(\lambda)$.

Proposition 4.12. Let $\lambda \in P^+$ and $a \in \mathbb{C}^\times$. Then $V_\xi(\lambda, a)$ is the irreducible quotient of $\overline{V_q(\lambda, a)}$. In particular, $V_\xi(\lambda, a) \cong V_\xi(\overline{\boldsymbol{\lambda}})$ where $\boldsymbol{\lambda}$ is as in (4.9).

Proof. Let v and L be as in Theorem 4.3. It follows that $L = U_{\mathbb{A}}(\mathfrak{g})v$ and, therefore, the irreducible quotient of $\overline{V_q(\lambda, a)}$ is isomorphic to $V_\xi(\lambda)$ as a $U_\xi(\mathfrak{g})$-module. The result now follows easily. $\square$

The following Proposition was proved in [1, 17] for $\xi \neq 1$ (for $\xi = 1$ the result follows from Theorem 4.10).

Proposition 4.13. Let $\xi \in \mathbb{C}'\backslash\{q\}$, $\boldsymbol{\lambda} \in \mathcal{P}^+_\xi$ and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. Then, $V_\xi(\boldsymbol{\lambda}) \cong V_\xi(\lambda)$ as $U_\xi(\mathfrak{g})$-module iff $\boldsymbol{\lambda}$ is as in (4.9) or (4.10) with ξ in place of q. $\square$

We now give explicit formulas for the action of the elements of $U_\xi(\tilde{\mathfrak{g}})$ on evaluation modules in the case $\mathfrak{g} = \mathfrak{sl}_2$. In this case one can normalize ev so that $m(\lambda) = 0$ for all $\lambda \in P^+$ and, hence, $V_q(\lambda, a) \cong V_q(\boldsymbol{\omega}_{i,a,\lambda(h_i)})$ where $I = \{i\}$. Given $\lambda \in P^+$, let v_0^λ be a highest-weight vector of $W_\xi(\lambda)$ and set $v_k^\lambda = (x_i^-)^{(k)} v_{k-1}^\lambda$, for $0 < k \leq \lambda(h_i)$. Then, $\{v_k^\lambda : 0 \leq k \leq \lambda(h_i)\}$ is a basis of $W_\xi(\lambda)$. For convenience, set $v_{-1}^\lambda = v_{\lambda(h_i)+1}^\lambda = 0$. The next lemma was proved in [14, §4.2].

Lemma 4.14. The following holds in $W_\xi(\lambda, a)$:

$$x_{i,r}^+ v_k^\lambda = (a\xi^{\lambda(h_i)-2k+2})^r [m-k+1]_\xi \; v_{k-1}^\lambda, \qquad x_{i,r}^- v_k^\lambda = (a\xi^{\lambda(h_i)-2k})^r [k+1]_\xi \; v_{k+1}^\lambda,$$

and

$$\psi_s^+ v_0^\lambda = (\xi - \xi^{-1})(a\xi^{\lambda(h_i)})^s [\lambda(h_i)]_\xi \; v_0^\lambda,$$

for every $r, s \in \mathbb{Z}, s > 0$. $\square$

4.3. **The Frobenius homomorphism.** Recall the Frobenius homomorphism defined in [40] (see also [19]).

Theorem 4.15. There exists a Hopf algebra homomorphism $\widetilde{\mathrm{Fr}}_\zeta : U_\zeta(\tilde{\mathfrak{g}}) \to U(\tilde{\mathfrak{g}})$ such that

$$\widetilde{\mathrm{Fr}}_\zeta(k_i) = 1, \qquad \widetilde{\mathrm{Fr}}_\zeta((x_{i,r}^{\pm})^{(k)}) = \begin{cases} \frac{(x_{i,r}^{\pm})^{k/l}}{(k/l)!}, & \text{if } l \text{ divides } k, \\ 0, & \text{otherwise.} \end{cases}$$

In particular,

$$\widetilde{\mathrm{Fr}}_\zeta(\Lambda_{i,r}) = \begin{cases} \Lambda_{i,r/l}, & \text{if } l \text{ divides } r, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mathrm{Fr}}_\zeta(\left[\begin{smallmatrix} k_i \\ r \end{smallmatrix}\right]) = \begin{cases} \binom{h_i}{r/l}, & \text{if } l \text{ divides } r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$ $\square$

Above, $\binom{h_i}{s} = \frac{h_i(h_i-1)\cdots(h_i-s+1)}{s!}$ for $s \geq 0$.

We shall denote by Fr_ζ the restriction of $\widetilde{\mathrm{Fr}}_\zeta$ to $U_\zeta(\mathfrak{g})$. If V is a $\tilde{\mathfrak{g}}$-module (resp. $\mathfrak{g}$-module), denote by $\widetilde{\mathrm{Fr}}_\zeta^*(V)$ (resp. $\mathrm{Fr}_\zeta^*(V)$) the pull-back of V by $\widetilde{\mathrm{Fr}}_\zeta$ (resp. Fr_ζ). Since $\widetilde{\mathrm{Fr}}_\zeta$ is a Hopf algebra map we have

$$\widetilde{\mathrm{Fr}}_\zeta^*(V \otimes W) \cong \widetilde{\mathrm{Fr}}_\zeta^*(V) \otimes \widetilde{\mathrm{Fr}}_\zeta^*(W). \tag{4.12}$$

Given $\lambda \in P^+$, there exist unique $\lambda' \in P_l^+, \lambda'' \in P^+$ such that $\lambda = \lambda' + l\lambda''$. Also, for $\boldsymbol{\lambda} \in \mathcal{P}^+$, recall the decomposition $\boldsymbol{\lambda} = \boldsymbol{\lambda}'\phi_l(\boldsymbol{\lambda}'')$ given by (2.11). The following theorem was proved in [19, 38].

Theorem 4.16. Let $\lambda \in \mathcal{P}^+$ and $\boldsymbol{\lambda} \in \mathcal{P}^+$. Then:

(a) $V_\zeta(\lambda) \cong V_\zeta(\lambda') \otimes V_\zeta(l\lambda'')$. Moreover, $V_\zeta(l\lambda'') \cong \mathrm{Fr}_\zeta^*(V(\lambda''))$.
(b) $V_\zeta(\boldsymbol{\lambda}) \cong V_\zeta(\boldsymbol{\lambda}') \otimes V_\zeta(\phi_l(\boldsymbol{\lambda}'')) \cong V_\zeta(\phi_l(\boldsymbol{\lambda}'')) \otimes V_\zeta(\boldsymbol{\lambda}')$. Moreover, $V_\zeta(\phi_l(\boldsymbol{\lambda}'')) \cong \widetilde{\mathrm{Fr}}_\zeta^*(V(\boldsymbol{\lambda}''))$. $\square$

Corollary 4.17. Let $\lambda \in P^+, a \in \mathbb{C}^\times$, and suppose $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then, $V_\zeta(l\lambda, a) \cong \widetilde{\mathrm{Fr}}_\zeta^*(V(\lambda, a^l)) \cong V_\zeta(\phi_l(\boldsymbol{\omega}_{\lambda,a^l}))$.

Proof. Using Proposition 4.12 and (2.12), one easily sees that $V_\zeta(l\lambda, a) \cong V_\zeta(\phi_l(\boldsymbol{\omega}_{\lambda,a^l}))$ and we are done by Theorems 4.16 and 4.10. $\square$

The following result is easily established.

Proposition 4.18. Let $\lambda, \mu \in P^+, a \in \mathbb{C}^\times$, and write $V(\lambda) \otimes V(\mu) = \bigoplus_{j=1}^{m} V(\nu_j)$ for some $m \in \mathbb{Z}_{\geq 0}$ and $\nu_j \in P^+$. Then $V(\lambda, a) \otimes V(\mu, a) \cong \bigoplus_{j=1}^{m} V(\nu_j, a)$. $\square$

By combining the above proposition with Theorem 4.10 we get:

Corollary 4.19. Let $m \in \mathbb{Z}_{\geq 0}, \lambda_j \in P^+$, and $a_j \in \mathbb{C}^\times$ for $j = 1, \ldots, m$. Then $\bigotimes_{j=1}^{m} V(\lambda_j, a_j)$ is irreducible iff $a_i \neq a_j$ for all $i \neq j$. $\square$

Corollary 4.20. Let $m \in \mathbb{Z}_{>0}, \lambda, \lambda_j \in P^+, a, a_j \in \mathbb{C}^\times$ for $j = 1, \ldots, m$ and $\boldsymbol{\lambda} = \prod_{j=1}^{m} \boldsymbol{\omega}_{\lambda_j,a_j}$. We have:

(a) If $a_i \neq a_j$ for all $i \neq j$, then $V_\zeta(\phi_l(\boldsymbol{\lambda})) \cong \bigotimes_{j=1}^{m} V_\zeta(\phi_l(\boldsymbol{\omega}_{\lambda_j,a_j}))$.
(b) If $V(\lambda) \otimes V(\lambda_1) \cong \bigoplus_{k=1}^{N} V(\mu_j)$ for some $N \in \mathbb{Z}_{>0}$ and $\mu_j \in P^+$, then $V_\zeta(\phi_l(\boldsymbol{\omega}_{\lambda,a})) \otimes V_\zeta(\phi_l(\boldsymbol{\omega}_{\lambda_1,a})) \cong \bigoplus_{k=1}^{N} V_\zeta(\phi_l(\boldsymbol{\omega}_{\mu_j,a}))$.

Proof. Immediate from Proposition 4.18, Theorems 4.16 and 4.10, and Corollary 4.17. □

The following theorem was proved in [14, 19].

Theorem 4.21. Let $\mathfrak{g} = \mathfrak{sl}_2, m, m' \in \mathbb{Z}_{\geq 0}, a_1, \dots, a_m, b_1, \dots, b_{m'} \in \mathbb{C}^\times, \lambda_1, \dots, \lambda_m \in P_l^+ \backslash \{0\}$, and $\mu_1, \dots, \mu_{m'} \in P^+ \backslash \{0\}$. Then, the tensor product

$$V_\xi(\lambda_1, a_1) \otimes \cdots \otimes V_\xi(\lambda_m, a_m) \otimes V_\xi(l\mu_1, b_1) \otimes \cdots \otimes V_\xi(l\mu_{m'}, b_{m'})$$

is irreducible iff the polynomials $\boldsymbol{\omega}_{i,a_j,\lambda_j(h_i)}$ and $\phi_l(\boldsymbol{\omega}_{\mu_k, b_k^l}), j = 1, \dots, m, k = 1, \dots m'$, are in general position. In particular, every simple object of $\widetilde{\mathcal{C}}_\xi$ is isomorphic to a tensor product of evaluation modules. □

4.4. **Highest-ℓ-weight tensor products.** The goal of this subsection is to prove a root of unity analogue of the following theorem which is the main result of [8].

Theorem 4.22. Suppose ξ is not a root of unity, $m \in \mathbb{Z}_{>0}$, and $\boldsymbol{\lambda}_j \in \mathcal{P}_\xi^+$ for $j = 1, \dots, m$. If $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m)$ is in ξ-resonant order, then $V_\xi(\boldsymbol{\lambda}_1) \otimes \cdots \otimes V_\xi(\boldsymbol{\lambda}_m)$ is a highest-ℓ-weight $U_\xi(\tilde{\mathfrak{g}})$-module.□

Corollary 4.23. Let ξ have infinite order. Given $a_1, \dots, a_k \in \mathbb{C}(\xi)$, there exists $\sigma \in S_k$ such that $V_\xi(\boldsymbol{\omega}_{i_1, a_{\sigma(1)}, m_1}) \otimes \cdots \otimes V_\xi(\boldsymbol{\omega}_{i_k, a_{\sigma(k)}, m_k})$ is highest-ℓ-weight for any choice of $i_1, \dots, i_k$ and of $m_1, \dots, m_k$.

Proof. Immediate from Theorem 4.22 and Lemma 2.4. □

As an application of this corollary alongside results of [5], the following theorem was proved in [12].

Theorem 4.24. Suppose ξ has infinite order and that $\boldsymbol{\lambda} = \prod_{j=1}^k \boldsymbol{\omega}_{i_j, a_j} \in \mathcal{P}_\xi^+$ for some $i_j \in I, a_j \in \mathbb{C}(\xi)^\times$. Then, there exists $\sigma \in S_k$ such that $W_\xi(\boldsymbol{\lambda}) \cong V_\xi(\boldsymbol{\omega}_{i_{\sigma(1)}, a_{\sigma(1)}}) \otimes \cdots \otimes V_\xi(\boldsymbol{\omega}_{i_{\sigma(k)}, a_{\sigma(k)}})$. In other words, if $\boldsymbol{\lambda}_i(u)$ is either constant or splits for all $i \in I$, then the associated Weyl module can be realized as a tensor product of fundamental representations. □

We now give a counterexample showing that both Corollary 4.23 and Theorem 4.24 do not hold in the case $\xi = \zeta$.

Example 4.25. Let $\mathfrak{g} = \mathfrak{sl}_2, I = \{i\}, \xi = \zeta, l = 3$, and suppose $a_1, a_2, a_3 \in \mathbb{C}^\times$ are such that $\{a_j/a_k : 1 \leq j < k \leq 3\} = \{\zeta, \zeta^2\}$. Let also $\boldsymbol{\lambda} = \prod_{j=1}^3 \boldsymbol{\omega}_{i,a_j}$. We claim that $W_\zeta(\boldsymbol{\lambda})$ is not isomorphic to $V_\zeta(\boldsymbol{\omega}_{i,a_{\sigma(1)}}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a_{\sigma(2)}}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a_{\sigma(3)}})$ for any $\sigma \in S_3$. For proving this, it suffices to show that $V_\zeta(\boldsymbol{\omega}_{i,a}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2})$ and $V_\zeta(\boldsymbol{\omega}_{i,a}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta})$ are not highest-ℓ-weight for any $a \in \mathbb{C}^\times$. Let v_1, v_2, v_3 be highest-ℓ-weight vectors for each factor of these tensor products, $v = v_1 \otimes v_2 \otimes v_3$, and $W = U_\zeta(\tilde{\mathfrak{g}})v$. It suffices to check that the dimension of W_{ω_i} is less than 3. In fact, one easily checks using Lemma 4.14 and Proposition 1.6(c) that for the later tensor product W_{ω_i} is 1-dimensional while for the former W_{ω_i} is 2-dimensional.

With a little more work one can compute composition series for the above tensor products. Namely, if $V = V_\zeta(\boldsymbol{\omega}_{i,a}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2})$, then $V \cong W \oplus V_\zeta(\boldsymbol{\omega}_{i,a\zeta})$ and there exists a non-split short exact sequence

$$0 \to V_\zeta(\boldsymbol{\omega}_{i,a}) \oplus V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2}) \to W \to V_\zeta(\boldsymbol{\lambda}) \to 0.$$

On the other hand, if $V = V_\zeta(\boldsymbol{\omega}_{i,a}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2}) \otimes V_\zeta(\boldsymbol{\omega}_{i,a\zeta})$, then $V \cong W \oplus V_\zeta(\boldsymbol{\omega}_{i,a\zeta}) \oplus V_\zeta(\boldsymbol{\omega}_{i,a\zeta^2})$ and there exists a non-split short exact sequence

$$0 \to V_\zeta(\boldsymbol{\omega}_{i,a}) \to W \to V_\zeta(\boldsymbol{\lambda}) \to 0.$$

In this case $W \cong W_\zeta(3\omega_i, a)$. □

Definition 4.26. Let $\mathfrak{g} = \mathfrak{sl}_2, I = \{i\}$, and $\boldsymbol{\lambda} = \prod_{j=1}^{k} \boldsymbol{\omega}_{i,a_k}$ for some $k \in \mathbb{Z}_{>0}$ and $a_j \in \mathbb{C}^\times$ for $j = 1, \ldots, k$. We say that $\boldsymbol{\lambda}$ is ζ-regular if there exists $\sigma \in S_k$ such that $W_\zeta(\boldsymbol{\lambda}) \cong V_\zeta(\boldsymbol{\omega}_{i,a_{\sigma(1)}}) \otimes \cdots \otimes V_\zeta(\boldsymbol{\omega}_{i,a_{\sigma(k)}})$. In general, given $\mathfrak{g}$, let $w_0 = s_{i_N} \cdots s_{i_1}$ be a reduced expression for w_0. We say that $\boldsymbol{\lambda} \in \mathcal{P}^+$ is ζ-regular (with respect to this reduced expression of w_0) if $(T_{i_{j-1}} \cdots T_{i_1} \boldsymbol{\lambda})_{i_j}$ is ζ_{i_j}-regular for all $j = 1, \ldots, N$.

Example 4.27. Let $\mathfrak{g} = \mathfrak{sl}_3, I = \{1, 2\}, l = 3, a \in \mathbb{C}^\times, \boldsymbol{\lambda} = \boldsymbol{\omega}_{1,a\zeta,2}\boldsymbol{\omega}_{2,a\zeta^2}$, and $w_0 = s_1 s_2 s_1$. Then $T_1\boldsymbol{\lambda} = (\boldsymbol{\omega}_{1,a,2})^{-1}\boldsymbol{\omega}_{2,a,3}$ and, hence, by Example 4.25, $\boldsymbol{\lambda}$ is not ζ-regular.

For the next two lemmas we will use the following notation. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_\xi^+, \lambda = \mathrm{wt}(\boldsymbol{\lambda}), \mu = \mathrm{wt}(\boldsymbol{\mu})$, and $v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\mu}}$ be highest-ℓ-weight vectors of highest-ℓ-weight modules V, W of highest ℓ-weight $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. Let also $v_{\boldsymbol{\lambda}}^w$ be a generator of $V_{w\lambda}$ for $w \in \mathcal{W}$.

Lemma 4.28. Suppose $i \in I$ and $w \in \mathcal{W}$ are such that $\ell(s_i w) = \ell(w) + 1$. Then, the $U_\xi(\tilde{\mathfrak{g}}_i)$-module $U_\xi(\tilde{\mathfrak{g}}_i)(v_{\boldsymbol{\lambda}}^w \otimes v_{\boldsymbol{\mu}})$ is a quotient of $W_\xi((T_w\boldsymbol{\lambda})_i \boldsymbol{\mu}_i)$.

Proof. Straightforward using Proposition 1.6 and Corollary 4.7 (cf. [8, Lemma 4.3] for ξ not a root of unity). □

Lemma 4.29. Suppose $v_{\boldsymbol{\lambda}}^{w_0} \otimes v_{\boldsymbol{\mu}} \in U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. Then, $V \otimes W = U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. Moreover, if $\xi = \zeta, V = U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})v_{\boldsymbol{\lambda}}$, $W = U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})v_{\boldsymbol{\mu}}$, and $v_{\boldsymbol{\lambda}} \in U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{n}}^+)v_{\boldsymbol{\lambda}}^{w_0}$, then $V \otimes W = U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$.

Proof. The proof is a refinement of that of [8, Lemma 4.2]. Since $(x_{i,r}^-)^{(k)} v_{\boldsymbol{\lambda}}^{w_0} = 0$ for all $i \in I, r, k \in \mathbb{Z}, k > 0$, it follows from Proposition 1.6 that $(x_{i,r}^-)^{(k)}(v_{\boldsymbol{\lambda}}^{w_0} \otimes v_{\boldsymbol{\mu}}) = v_{\boldsymbol{\lambda}}^{w_0} \otimes ((x_{i,r}^-)^{(k)} v_{\boldsymbol{\mu}})$. Hence, $v_{\boldsymbol{\lambda}}^{w_0} \otimes W \subseteq U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. Similarly, if $\xi = \zeta, V = U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})v_{\boldsymbol{\lambda}}$, and $W = U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})v_{\boldsymbol{\mu}}$, then $v_{\boldsymbol{\lambda}}^{w_0} \otimes W \subseteq U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$.

Next we prove that $v_{\boldsymbol{\lambda}} \otimes W \subseteq U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$ and, under the hypothesis of the "moreover part" of the statement, that $v_{\boldsymbol{\lambda}} \otimes W \subseteq U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. Since $v_{\boldsymbol{\lambda}} \in U_\xi(\tilde{\mathfrak{n}}^+)v_{\boldsymbol{\lambda}}^{w_0}$ (respectively, $v_{\boldsymbol{\lambda}} \in U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{n}}^+)v_{\boldsymbol{\lambda}}^{w_0}$), it suffices to show that

$$\left((x_{i_1,r_1}^+)^{(k_1)} \cdots (x_{i_m,r_m}^+)^{(k_m)}\, v_{\boldsymbol{\lambda}}^{w_0}\right) \otimes W \subseteq U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}}) \tag{4.13}$$

for all $m \in \mathbb{Z}_{\geq 0}, k_j \in \mathbb{Z}_{>0}, i_j \in I, r_j \in \mathbb{Z}$ (for the "moreover part" it suffices to prove this with $k_j = 1$ for all $j = 1, \ldots, m$ and with $U_\zeta^{\mathrm{fin}}(\tilde{\mathfrak{g}})$ in place of $U_\xi(\tilde{\mathfrak{g}})$). We proceed by induction on the height of $\sum_j k_j \alpha_{i_j}$. Induction clearly starts for $m = 0$. We shall write the proof for the first statement only since the second one is proved similarly.

Let $v = (x_{i_2,r_2}^+)^{(k_2)} \cdots (x_{i_m,r_m}^+)^{(k_m)}\, v_{\boldsymbol{\lambda}}^{w_0}$ and assume, by induction hypothesis, that $v \otimes W \subseteq U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. We now prove that $(x_{i_1,r_1}^+ v) \otimes W \subseteq U_\xi(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. We consider only the case $r_1 \geq 0$ since the case $r_1 < 0$ is similar. The proof is by induction on k_1 with a further subinduction on r_1. Let $v' \in W$. By Proposition 1.6 we have

$$(x_{i_1,r_1}^+)^{(k_1)}(v \otimes v') = ((x_{i_1,r_1}^+)^{(k_1)} v) \otimes v' + \varpi + \varpi'$$

where ϖ is a sum of vectors which belong to

$$\left((x_{i'_1,r'_1}^+)^{(k'_1)} \cdots (x_{i'_{m'},r'_{m'}}^+)^{(k'_{m'})}\, v_{\boldsymbol{\lambda}}^{w_0}\right) \otimes W$$

with the height of $\sum_{j=1}^{m'} k'_j\alpha_{i'_j}$ strictly smaller than that of $\sum_{j=1}^{m} k_j\alpha_{i_j}$, while ϖ' is a sum of vectors belonging to

$$\left((x^+_{i_1,r_1})^{(k)}(x^+_{i_1,s_1})^{(k'_1)}\cdots(x^+_{i_1,s_{m'}})^{(k'_{m'})}\ v\right)\otimes W$$

with $0\le k<k_1, 0\le s_j<r_1$ and $k+\sum_{j=1}^{m'}k'_j=k_1$. Hence, by the induction hypothesis on the height we have $\varpi\in U_\xi(\tilde{\mathfrak g})(v_{\boldsymbol\lambda}\otimes v_{\boldsymbol\mu})$, while the same is true for ϖ' by the subinduction hypothesis on k_1 and r_1 (observe that if $k_1=1$ then $k=0$ and if $r_1=0$ then $\varpi'=0$ which shows that both subinductions start). Since $(x^+_{i_1,r_1})^{(k_1)}(v\otimes v')$ obviously belong to $U_\xi(\tilde{\mathfrak g})(v_{\boldsymbol\lambda}\otimes v_{\boldsymbol\mu})$, it follows that $((x^+_{i_1,r_1})^{(k_1)}v)\otimes v'\in U_\xi(\tilde{\mathfrak g})(v_{\boldsymbol\lambda}\otimes v_{\boldsymbol\mu})$ for all $v'\in W$. This completes the proof of (4.13).

Finally, it remains to show that

$$\left((x^-_{i_1,r_1})^{(k_1)}\cdots(x^-_{i_m,r_m})^{(k_m)}\ v_{\boldsymbol\lambda}\right)\otimes W\subseteq U_\xi(\tilde{\mathfrak g})(v_{\boldsymbol\lambda}\otimes v_{\boldsymbol\mu})\tag{4.14}$$

for all $m\in\mathbb Z_{\ge0}, k_j\in\mathbb Z_{>0}, i_j\in I, r_j\in\mathbb Z$. The proof of (4.14) is identical to the proof of (4.13) replacing $(x^+_{i_j,r_j})^{(k_j)}$ by $(x^-_{i_j,r_j})^{(k_j)}$ and $v^{w_0}_{\boldsymbol\lambda}$ by $v_{\boldsymbol\lambda}$. ☐

Proposition 4.30. Let $\mathfrak g=\mathfrak{sl}_2, m\in\mathbb Z_{>0}$, and $\boldsymbol\lambda_j\in\mathcal P^+_\xi$ for $j=1,\dots,m$. If $(\boldsymbol\lambda_1,\dots,\boldsymbol\lambda_m)$ is in ξ-resonant order, then $V_\xi(\boldsymbol\lambda_1)\otimes\cdots\otimes V_\xi(\boldsymbol\lambda_m)$ is a highest-ℓ-weight $U_\xi(\tilde{\mathfrak g})$-module. Moreover, if $\xi=\zeta$ and $\boldsymbol\lambda_j\in\mathcal P^+_l$ for all $j=1,\dots,m$, then $V_\zeta(\boldsymbol\lambda_1)\otimes\cdots\otimes V_\zeta(\boldsymbol\lambda_m)$ is generated by the action of $U^{\rm fin}_\zeta(\tilde{\mathfrak g})$ on the top weight space.

Proof. If $\xi=1$ this is part of Theorem 4.10 and if ξ has infinite order this follows from [14, Lemma 4.10] which will be reviewed below. Thus, assume $\xi=\zeta$. We first prove the "moreover" part. We proceed by induction on m observing that there is nothing to prove when $m=1$. Let $v_j, j=1,\dots,m$, be highest-ℓ-weight vectors for $V_\zeta(\boldsymbol\lambda_j)$ and $v'=v_2\otimes\cdots\otimes v_m$. By the induction hypothesis, $V'=U^{\rm fin}_\zeta(\tilde{\mathfrak g})v'$.

The proof runs parallel to that of [14, Lemma 4.10]. Let $I=\{i\}$. By Theorem 4.21, it suffices to consider the case $\boldsymbol\lambda_j=\boldsymbol\omega_{i,a_j,n_j}$ for all $j=1,\dots,m$ and some $a_j\in\mathbb C^\times$ and $0<n_j<l$. Given $k\in\{0,\dots,n_j\}$, set $v_j^k=(x_i^-)^kv_j$ so that $\{v_j^k:k=0,\dots,n_1\}$ is a basis of $V(\boldsymbol\lambda_j)$. By Lemma 4.29, it suffices to show that

$$v_1^{k+1}\otimes v'\in U^{\rm fin}_\zeta(\tilde{\mathfrak g})(v_1^k\otimes v')\quad\text{for all}\quad k=0,\dots,n_1-1.\tag{4.15}$$

It is not difficult to see, using Proposition 1.6, that there exist $a_{r,s}\in\mathbb C$ such that

$$x^-_{i,r}(v_1^k\otimes v')=\sum_{s=1}^m a_{r,s}w_s\tag{4.16}$$

where $w_1=v_1^{k+1}\otimes v', w_2=v_1^k\otimes v_2^1\otimes v_3\otimes\cdots\otimes v_m,\dots,w_m=v_1^k\otimes v_2\otimes\cdots\otimes v_{m-1}\otimes v_m^1$. In fact, $a_{r,s}$ can be explicitly computed using Lemma 4.14. It turns out that the determinant of the matrix $(a_{r,s}), 0\le r,s\le m$, is different from zero iff $a_1/a_j\ne\zeta^{-(n_1+n_j-2k)}$ and $a_j/a_\ell\ne\zeta^{-(n_j+n_\ell)}$ for all $1<j<\ell\le m$ (see the proof of [14, Lemma 4.10]). Since $(\boldsymbol\lambda_1,\dots,\boldsymbol\lambda_m)$ is in ξ-resonant order, (4.15) follows.

To prove the first statement, write $\boldsymbol\lambda_j=\boldsymbol\lambda'_j\phi_l(\boldsymbol\lambda''_j)$ as in (2.11) and notice that Theorem 4.16(b) and Corollary 4.20(a) imply that

$$V_\zeta(\boldsymbol\lambda_1)\otimes\cdots\otimes V_\zeta(\boldsymbol\lambda_m)\cong V_\zeta(\boldsymbol\lambda'_1)\otimes\cdots\otimes V_\zeta(\boldsymbol\lambda'_m)\otimes V_\zeta(\phi_l(\boldsymbol\mu))$$

where $\boldsymbol\mu=\prod_{j=1}^m\boldsymbol\lambda''_j$. Set $V=V_\zeta(\boldsymbol\lambda'_1)\otimes\cdots\otimes V_\zeta(\boldsymbol\lambda'_m)$ and $\boldsymbol\lambda=\prod_{j=1}^m\boldsymbol\lambda'_j$. Since we have already proved the "moreover" part of the theorem, it follows that V is a highest-ℓ-weight module of highest-ℓ-weight $\boldsymbol\lambda$. Let $v_{\boldsymbol\lambda}, v_{\boldsymbol\mu}$ be highest-ℓ-weight vectors of V and $W=V_\zeta(\phi_l(\boldsymbol\mu))$, respectively. By the "moreover"

part of Theorem 4.16(b), $x_{i,r}^-$ acts trivially on W and, hence, $V \otimes v_{\boldsymbol{\mu}} \in U_\zeta^{\rm fin}(\tilde{\mathfrak{g}})(v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}})$. We are done by Lemma 4.29. $\square$

We now prove a stronger version of Proposition 4.30 for two-fold tensor products of evaluation representations. For this purpose only, if $l = 1$, i.e., if ξ is generic, set $P_1^+ = P^+$ (recall that $P_1^+ = \{0\}$ in our usual notation). Thus, let $\mathfrak{g} = \mathfrak{sl}_2$, $I = \{i\}$, $a, b \in \mathbb{C}(\xi)^\times$, and $\lambda, \mu \in P_l^+$ be such that $a/b \neq \xi^{-(\lambda(h_i)+\mu(h_i)-2k)}$ for any $0 \le k < \min\{\lambda(h_i), \mu(h_i)\}$. In other words, $(\boldsymbol{\omega}_{i,a,\lambda(h_i)}, \boldsymbol{\omega}_{i,b,\mu(h_i)})$ is in weak ξ-resonant order. Observe that if $\lambda(h_i) \le \mu(h_i)$, then $(\boldsymbol{\omega}_{i,a,\lambda(h_i)}, \boldsymbol{\omega}_{i,b,\mu(h_i)})$ is in ξ-resonant order as well, but this may not be the case otherwise.

Proposition 4.31. With the above hypotheses, $V_\xi(\lambda, a) \otimes V_\xi(\mu, b)$ is a highest ℓ-weight module.

Proof. We use the notation of Lemma 4.14. Namely, let v_0^λ, v_0^μ be highest-weight vectors of $V_\xi(\lambda)$ and $V_\xi(\mu)$, respectively, and set $v_k^\nu = (x_i^-)^{(k)} v_{k-1}^\nu, \nu = \lambda, \mu$. We first prove that $v_k^\lambda \otimes v_0^\mu, v_{k-1}^\lambda \otimes v_1^\mu \in U_\xi^{\rm fin}(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$ for $0 \le k \le \min\{\lambda(h_i), \mu(h_i)\}$ by induction on k. This is done similarly to the proof of [14, Lemma 4.10]. Since in the present situation that proof simplifies significantly, let us be precise. Induction obviously starts when $k = 0$. Hence, assume $0 \le k < \min\{\lambda(h_i), \mu(h_i)\}$ is such that $v_k^\lambda \otimes v_0^\mu \in U_\xi^{\rm fin}(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$ and let $a_{j,m} \in \mathbb{C}(\xi), j, m \in \{1, 2\}$ be such that

$$x_{i,r}^-(v_k^\lambda \otimes v_0^\mu) = a_{r,1}\ v_{k+1}^\lambda \otimes v_0^\mu + a_{r,2}\ v_k^\lambda \otimes v_1^\mu,$$

for $r = 1, 2$. It suffice to show that $\det(a_{j,m}) \neq 0$. Using Proposition 1.6 and Lemma 4.14 one computes:

$$(a_{j,m}) = \begin{pmatrix} [k+1]_\xi a \xi^{\lambda(h_i)+\mu(h_i)-2k} & b\xi^{\mu(h_i)} \\ [k+1]_\xi a \xi^{\lambda(h_i)+\mu(h_i)-2k}(a\xi^{\lambda(h_i)-2k} + b\left(\xi^{\mu(h_i)} - \xi^{-\mu(h_i)}\right)) & (b\xi^{\mu(h_i)})^2 \end{pmatrix}.$$

Therefore,

$$\det(a_{j,m}) = [k+1]_\xi ab\xi^{\lambda(h_i)+2\mu(h_i)-2k}\left(b\xi^{-\mu(h_i)} - a\xi^{\lambda(h_i)-2k}\right)$$

which vanishes iff either $a/b = \xi^{-(\lambda(h_i)+\mu(h_i)-2k)}$ or $[k+1]_\xi = 0$. The former does not happen because $k < \min\{\lambda(h_i), \mu(h_i)\}$. As for the latter, if ξ is generic $[k+1]_\xi \neq 0$ regardless of the value of $k \ge 0$, while for $\xi = \zeta$ this follows since $k < \min\{\lambda(h_i), \mu(h_i)\} < l$. The induction step is proved.

If $\lambda(h_i) \le \mu(h_i)$ (which is assumed in the proof of [14, Lemma 4.10]), the proof is completed by using Lemma 4.29. Otherwise, let $V = V_\xi(\lambda, a) \otimes V_\xi(\mu, b)$. By Lemma 4.29 again, it suffices to show that $v_{\lambda(h_i)}^\lambda \otimes v_0^\mu \in U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$. To prove this, in turn, by Proposition 3.2, it suffices to show that

$$V_{\lambda+\mu-\mu(h_i)\alpha_i} \subseteq U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$$

since $v_{\lambda(h_i)}^\lambda \otimes v_0^\mu \in V_{\lambda+\mu-\lambda(h_i)\alpha_i}$ and $\lambda + \mu - \lambda(h_i)\alpha_i = w_0(\lambda + \mu - \mu(h_i)\alpha_i)$. We now use the first part of the proof to show that

$$V_{\lambda+\mu-k\alpha_i} \subseteq U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu) \qquad \text{for all } 0 \le k \le \mu(h_i). \tag{4.17}$$

Observe that the vectors $v_j := v_{k-j}^\lambda \otimes v_j^\mu, j = 0, \ldots, k$, form a basis of $V_{\lambda+\mu-k\alpha_i}$. The proof of (4.17) is again done by induction on k. Since for $k = 0$ there is nothing to prove, assume $0 \le k < \mu(h_i)$ is such that (4.17) holds. This implies $v_j \in U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$ for all $0 \le j \le k$. Using Proposition 1.6 and Lemma 4.14 once more we get

$$x_i^- v_j = [k-j+1]_\xi \xi^{\mu(h_i)-2j}(v_{k-j+1}^\lambda \otimes v_j^\mu) + [j+1]_\xi(v_{k-j}^\lambda \otimes v_{j+1}^\mu) \ \in\ U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu).$$

Since $[j+1]_\xi \neq 0$ for all $j = 0, \ldots, k$ and, by the first part of the proof, $v_{k+1}^\lambda \otimes v_0^\mu \in U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$, one easily proves recursively on $j = 0, \ldots, k$, that $v_{k-j}^\lambda \otimes v_{j+1}^\mu \ \in\ U_\xi(\tilde{\mathfrak{g}})(v_0^\lambda \otimes v_0^\mu)$. $\square$

Remark. As of the moment we do not know if the above proposition remains true or not in greater generality. Thus, we pose the following question. Can "ξ-resonant order" be replaced by "weak ξ-resonant order" in Theorem 4.22? By looking at the proof of Theorem 4.22, one sees that it suffices to answer this for $\mathfrak{g} = \mathfrak{sl}_2$. In other words, it suffices to show a version of Proposition 4.31 for tensor products with arbitrary number of factors. The generalization of the first part of the proof of Proposition 4.31 is identical to the proof of [14, Lemma 4.10] (this is implicitly used in the proof of Theorem 4.22). Hence, it remains to generalize the argument of the second part of the proof.

The next lemma is easily established.

Lemma 4.32. Let $\boldsymbol{\lambda} \in \mathcal{P}_\xi^+$ and let v be a highest-ℓ-weight vector of $V_\xi(\boldsymbol{\lambda})$. Then $U_\xi(\tilde{\mathfrak{g}}_i)v$ is an irreducible $U_\xi(\tilde{\mathfrak{g}}_i)$-module for all $i \in I$. $\square$

Theorem 4.33. Let $m \in \mathbb{Z}_{>0}$ and $\boldsymbol{\lambda}_j \in \mathcal{P}^+$ for $j = 1, \dots, m$. If $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m)$ is in ζ-resonant order and $\boldsymbol{\lambda}_j$ is ζ-regular for all $j = 1, \dots, m-1$, then $V_\zeta(\boldsymbol{\lambda}_1) \otimes \cdots \otimes V_\zeta(\boldsymbol{\lambda}_m)$ is a highest-ℓ-weight $U_\zeta(\tilde{\mathfrak{g}})$-module.

Proof. The proof runs parallel to that of Theorem 4.22 with a few modifications. We again proceed by induction on m and observe that there is nothing to prove when $m = 1$. Once more, let $v_j, j = 1, \dots, m$ be highest-ℓ-weight vectors for $V_\zeta(\boldsymbol{\lambda}_j)$ and $v' = v_2 \otimes \cdots \otimes v_m$. By the induction hypothesis, $V' = U_\zeta^{\rm fin}(\tilde{\mathfrak{g}})v'$.

Fix a reduced expression for w_0, say $s_{i_N} \cdots s_{i_1}$. By Lemma 4.29, it suffices to show that

$$v_1^{s_{i_j}\cdots s_{i_1}} \otimes v' \in U_\zeta(\tilde{\mathfrak{g}}_{i_j})(v_1^{s_{i_{j-1}}\cdots s_{i_1}} \otimes v') \quad \text{for all} \quad j = 1, \dots, N. \tag{4.18}$$

Now, $U_\zeta(\tilde{\mathfrak{g}}_{i_j})v_k$ is an irreducible $U_\zeta(\tilde{\mathfrak{g}}_{i_j})$-module of highest-$\ell$-weight $(\boldsymbol{\lambda}_k)_{i_j}$ by Lemma 4.32. By hypothesis, the $(m-1)$-tuple $((\boldsymbol{\lambda}_2)_{i_j}, \dots, (\boldsymbol{\lambda}_m)_{i_j})$ is in ζ_{i_j}-resonant order. In particular, it follows from Lemma 4.32 and Proposition 4.30 that

$$U_\zeta(\tilde{\mathfrak{g}}_{i_j})v' = (U_\zeta(\tilde{\mathfrak{g}}_{i_j})v_2) \otimes \cdots \otimes (U_\zeta(\tilde{\mathfrak{g}}_{i_j})v_m).$$

On the other hand, $U_\zeta(\tilde{\mathfrak{g}}_{i_j})v_1^{s_{i_{j-1}}\cdots s_{i_1}}$ is a quotient of $W_\zeta((T_{i_{j-1}} \cdots T_{i_1}\boldsymbol{\lambda}_1)_{i_j})$ by Lemma 4.28. Since $\boldsymbol{\lambda}_1$ is ζ-regular, $W_\zeta((T_{i_{j-1}} \cdots T_{i_1}\boldsymbol{\lambda}_1)_{i_j})$ is isomorphic to a tensor product of the form $V_\zeta(\boldsymbol{\omega}_{i_j,a_1}) \otimes \cdots \otimes V_\zeta(\boldsymbol{\omega}_{i_j,a_k})$ for some $a_1, \dots, a_k \in \mathbb{C}^\times$ and where $k = \mathrm{wt}(T_{i_{j-1}} \cdots T_{i_1}\boldsymbol{\lambda}_1)(h_{i_j})$. The assumption that $\big((T_{i_{j-1}} \cdots T_{i_1}\boldsymbol{\lambda}_1)_{i_j}, (\boldsymbol{\lambda}_2)_{i_j}, \dots, (\boldsymbol{\lambda}_m)_{i_j}\big)$ is in ζ_{i_j}-resonant order implies that $(\boldsymbol{\omega}_{i_j,a_1}, \dots, \boldsymbol{\omega}_{i_j,a_k}, (\boldsymbol{\lambda}_2)_{i_j}, \dots, (\boldsymbol{\lambda}_m)_{i_j})$ is also in ζ_{i_j}-resonant order. It follows from Proposition 4.30 that

$$U_\zeta(\tilde{\mathfrak{g}}_{i_j})(v_1^{s_{i_{j-1}}\cdots s_{i_1}} \otimes v') = (U_\zeta(\tilde{\mathfrak{g}}_{i_j})v_1^{s_{i_{j-1}}\cdots s_{i_1}}) \otimes (U_\zeta(\tilde{\mathfrak{g}}_{i_j})v')$$

which implies (4.18). $\square$

We now give a counterexample showing that the "loop" analogue of the first statement of Proposition 4.1(a) does not hold in general.

Example 4.34. Let $\mathfrak{g} = \mathfrak{sl}_2, I = \{i\}$ and $a, b \in \mathbb{A}^\times$ be such that $\frac{a}{b} \neq q^{-2}$. It follows that $V = V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{i,b})$ is a highest-ℓ-weight module with highest ℓ-weight $\boldsymbol{\lambda} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,b} \in \mathcal{P}_\mathbb{A}^s$. Let v and w be highest-ℓ-weight vectors of $V_q(\boldsymbol{\omega}_{i,a})$ and $V_q(\boldsymbol{\omega}_{i,b})$, respectively, and set $v_1 = x_i^- v, w_1 = x_i^- w$. Then $v_1 \in V_q(\boldsymbol{\omega}_{i,a})_{\boldsymbol{\omega}_{i,aq^2}^{-1}}, w_1 \in V_q(\boldsymbol{\omega}_{i,b})_{\boldsymbol{\omega}_{i,bq^2}^{-1}}$, $v \otimes w_1 \in V_{\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,bq^2}^{-1}}$, and $v_1 \otimes w \in V_{\boldsymbol{\omega}_{i,b}\boldsymbol{\omega}_{i,aq^2}^{-1}}$.

Set also $\mathbf{v}_1 = x_i^-(v \otimes w), \mathbf{v}_2 = x_{i,1}^-(v \otimes w)$, and $L = U_\mathbb{A}(\tilde{\mathfrak{g}})(v \otimes w)$. It is known that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an $\mathbb{A}$-basis for the zero weight space $L_0 := L \cap V_0$. Using Lemma 4.14 and Proposition 1.6, one easily computes that

$$\mathbf{v}_1 = q^{-1}v_1 \otimes w + v \otimes w_1 \qquad \text{and} \qquad \mathbf{v}_2 = aq^2 v_1 \otimes w + bqv \otimes w_1. \tag{4.19}$$

We will show that there exists no $\mathbb{A}$-basis of L_0 formed by ℓ-weight vectors provided $(b - aq^2) \notin \mathbb{A}^\times$ and $a \neq b$ ($a = b$ is the only choice for the pair (a, b) so that any nonzero vector of V_0 is an ℓ-weight

vector). By contradiction, suppose there exist $\alpha, \beta \in \mathbb{C}(q)$ such that $\{\alpha v \otimes w_1, \beta v_1 \otimes w\}$ is an $\mathbb{A}$-basis of L_0. Then, the matrix A whose columns are the coordinates of $\mathbf{v}_1, \mathbf{v}_2$ with respect to this basis must have entries in $\mathbb{A}$ and its determinant must be in $\mathbb{A}^\times$. Using (4.19) one easily sees that

$$A = \begin{pmatrix} (q\alpha)^{-1} & aq^2\alpha^{-1} \\ \beta^{-1} & qb\beta^{-1} \end{pmatrix} \quad \text{and} \quad \det(A) = (\alpha\beta)^{-1}(b - aq^2). \tag{4.20}$$

It follows that $\alpha^{-1}, \beta^{-1} \in \mathbb{A}$ and, hence, $\det(A) \in \mathbb{A}^\times$ iff $\alpha, \beta, (b - aq^2) \in \mathbb{A}^\times$. Now, there are plenty of choices for $a, b \in \mathbb{A}^\times$ such that $b - aq^2 \notin \mathbb{A}^\times$.

4.5. **On Jordan-Hölder constituents.** We now prove several results regarding the irreducible constituents of $U_\xi(\tilde{\mathfrak{g}})$-modules obtained by specializing $U_q(\tilde{\mathfrak{g}})$-modules. We begin with the following general fact.

Proposition 4.35. Let $\mathcal{C}$ be a Jordan-Hölder tensor category and V, W objects in $\mathcal{C}$. The set of irreducible constituents of $V \otimes W$ (counted with multiplicities) is the union of the sets of irreducible constituents of $V_i \otimes W_j$ where V_i runs through the irreducible constituents of V and W_j runs through the irreducible constituents of W. □

Recall the definitions of $I_\bullet \subseteq I$ from §1.1 (Table 1) and of $\mathcal{P}^+_{\xi, I_\bullet}$ given at the end of §2.3.

Theorem 4.36. Every simple object of $\mathcal{C}_\xi$ is isomorphic to the quotient of a submodule of a tensor product of the modules $W_\xi(\omega_i)$ for $i \in I_\bullet$.

Proof. For simplicity, we write the proof of the theorem for the case that $I_\bullet$ is a singleton, i.e., that $\mathfrak{g}$ is not of type D_{2m}. Thus, let i denote the unique element of $I_\bullet$. For $\xi = 1$ the result is well-known and the proof for ξ not a root of unity is analogous. Moreover, every simple object of $\mathcal{C}_\xi$ is an irreducible summand of a tensor power of $V_\xi(\omega_i) = W_\xi(\omega_i)$. We now consider the case $\xi = \zeta$.

Let $\lambda \in P^+$ and $m \in \mathbb{Z}_{\geq 0}$ be such that $V_q(\lambda)$ is a summand of $V_q(\omega_i)^{\otimes m}$. Fix highest-weight vectors v_j of the j-th factor of this tensor product, let $L_j = U_{\mathbb{A}}(\mathfrak{g})v_j$, $j = 1, \ldots, m$, and $L = L_1 \otimes \cdots \otimes L_m$. Quite clearly $\overline{L} \cong \otimes_j \overline{L_j} \cong W_\zeta(\omega_i)^{\otimes m}$. Hence, it suffices to show that there exists $v \in L$ such that $x_k^+ v = 0$ for all $k \in I$, $U_q(\mathfrak{g})v \cong V_q(\lambda)$, and the image $\bar{v}$ of v in $\overline{L}$ is nonzero.

The $\mathbb{A}$-module L_λ has an $\mathbb{A}$-basis formed by elements of the form

$$\left((x^-_{i_{1,1}})^{(k_{1,1})} \cdots (x^-_{i_{r_1,1}})^{(k_{r_1,1})} v_1\right) \otimes \cdots \otimes \left((x^-_{i_{1,m}})^{(k_{1,m})} \cdots (x^-_{i_{r_m,m}})^{(k_{r_m,m})} v_m\right) := v_{\vec{i},\vec{k}}.$$

Here $\vec{i} = (i_{1,1}, \cdots, i_{r_1,1}; \cdots; i_{1,m}, \cdots, i_{r_m,m})$ and similarly for $\vec{k}$. It easily follows that there is an $\mathbb{A}$-linear combination of such vectors, say v', satisfying the two first properties required of v. Write

$$v' = \sum_{\vec{i},\vec{k}} c_{\vec{i},\vec{k}}\, v_{\vec{i},\vec{k}}, \quad c_{\vec{i},\vec{k}} \in \mathbb{A},$$

and observe that there exist $c'_{\vec{i},\vec{k}} \in \mathbb{C}[q]$ and $f \in \mathbb{A}$ such that the nonzero $c'_{\vec{i},\vec{k}}$ are relatively prime in $\mathbb{C}[q]$ and

$$v' = f \sum_{\vec{i},\vec{k}} c'_{\vec{i},\vec{k}}\, v_{\vec{i},\vec{k}} := fv.$$

Since the $c'_{\vec{i},\vec{k}}$ are relatively prime, it follows that $\epsilon_\zeta(c'_{\vec{i},\vec{k}}) \neq 0$ for at least one value of the pair $(\vec{i}, \vec{k})$. The theorem follows. □

In order to prove an affine analogue of the above theorem we will need the following proposition which will also be used to prove the main result of §4.6.

Proposition 4.37. Let $\boldsymbol{\lambda} \in \mathcal{P}^s_{\mathbb{A}}$, V a nontrivial quotient of $W_q(\boldsymbol{\lambda})$, and $\xi \in \mathbb{C}'\backslash\{q\}$. If $V_q(\boldsymbol{\mu})$ is an irreducible constituent of V with multiplicity m, then $V_\xi(\overline{\boldsymbol{\mu}})$ is an irreducible constituent of $\overline{V}$ with multiplicity at least m.

Proof. By an obvious induction on the length of V, it suffices to show the proposition in the case $V_q(\boldsymbol{\mu})$ is a submodule of V. The argument is essentially the same as that of [31, Proposition 4.16]. Namely, let v be a highest-ℓ-weight vector of V and $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then, from the proof of Theorem 4.3, L has an $\mathbb{A}$-basis consisting of vectors which are $\mathbb{A}$-linear combinations of elements of the form $(x^-_{\alpha_{i_1},r_1})^{(k_1)}\cdots(x^-_{\alpha_{i_s},r_s})^{(k_s)}v$. Let $v_1,\cdots,v_r$ be an $\mathbb{A}$-basis for L_μ where $\mu = \mathrm{wt}(\boldsymbol{\mu})$. Any highest-$\ell$-weight vector for $V_q(\boldsymbol{\mu})$ is a solution $\sum_{j=1}^r c_jv_j$, for some $c_j \in \mathbb{C}(q)$, of the linear system

$$(x^+_{i,s})^{(k)}\left(\sum_{j=1}^r c_jv_j\right) = 0, \qquad \Lambda_{i,s}\left(\sum_{j=1}^r c_jv_j\right) = \boldsymbol{\Psi\mu}(\Lambda_{i,s})\left(\sum_{j=1}^r c_jv_j\right)$$

for all $i \in I, s \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. By (induction) assumption, there exists a nontrivial solution for this system. Since L is admissible and the ℓ-weights of V are in $\mathcal{P}_{\mathbb{A}}$ (by Theorem 3.21 given that $\boldsymbol{\lambda} \in \mathcal{P}^s_{\mathbb{A}}$), it follows that there exists a solution with the c_j lying in $\mathbb{A}$. Hence, there is also a solution with $c_j \in \mathbb{C}[q]$ and such that the nonzero c_j are relatively prime. This completes the proof similarly to that of Theorem 4.36. □

Theorem 4.38. For every $\boldsymbol{\lambda} \in \mathcal{P}^+_\xi$, there exists $\boldsymbol{\mu} \in \mathcal{P}^+_{\xi,I_\bullet}$ such that $V_\xi(\boldsymbol{\lambda})$ is an irreducible constituent of $W_\xi(\boldsymbol{\mu})$. Moreover, if $\boldsymbol{\lambda} \in \mathcal{P}^s_{\mathbb{A}}$, then $\boldsymbol{\mu}$ can be chosen to be in $\mathcal{P}^s_{\mathbb{A}}$ as well.

Proof. For $\xi = q$, it follows from the results of [18, 22] that $V_q(\boldsymbol{\lambda})$ is a constituent of a tensor product of the form $V_q(\boldsymbol{\omega}_{i_1,a_1}) \otimes \cdots V_q(\boldsymbol{\omega}_{i_m,a_m})$ with $i_j \in I_\bullet$ and $a_j \in \mathbb{C}^\times$ ($a_j \in \mathbb{A}^\times$ if $\boldsymbol{\lambda} \in \mathcal{P}^s_{\mathbb{A}}$). The theorem now follows in this case by Corollaries 4.23 and 3.22.

For $\xi \in \mathbb{C}'$, consider the module $V_q(\boldsymbol{\lambda})$ and apply the theorem for the case $\xi = q$. Thus, let $\boldsymbol{\mu} \in \mathcal{P}^+_{\xi,I_\bullet} \cap \mathcal{P}^s_{\mathbb{A}}$ be such that $V_q(\boldsymbol{\lambda})$ is a constituent of $W_q(\boldsymbol{\mu})$. By Proposition 4.37, $V_\xi(\boldsymbol{\lambda})$ is a constituent of $\overline{W_q(\boldsymbol{\mu})}$. □

4.6. **Blocks.** Recall the notation introduced in §2.3. We shall need the following proposition which follows from the results of [12, 22].

Proposition 4.39. Let $\boldsymbol{\mu} \in \mathcal{P}^+_{q,I_\bullet}$ and $\boldsymbol{\lambda} = \boldsymbol{\mu\tau}_{q,k,a}$ for some $k \in \{1,2,3\}$ and some $a \in \mathbb{C}(q)^\times$. If V is an irreducible constituent of $W_q(\boldsymbol{\lambda})$, then V is an irreducible constituent of $W_q(\boldsymbol{\mu})$. □

An object $V \in \widetilde{\mathcal{C}}_\xi$ is said to have elliptic character $\gamma \in \widetilde{\Gamma}_\xi$ if $V_{\boldsymbol{\mu}} \neq 0$ implies $\gamma_\xi(\boldsymbol{\mu}) = \gamma$. Denote by $\widetilde{\mathcal{C}}^\gamma_\xi$ the abelian subcategory of $\widetilde{\mathcal{C}}_\xi$ consisting of representations with elliptic character γ. The main result of this section is the following theorem.

Theorem 4.40. The categories $\widetilde{\mathcal{C}}^\gamma_\xi, \gamma \in \widetilde{\Gamma}_\xi$, are the blocks of $\widetilde{\mathcal{C}}_\xi$.

This theorem was first proved in [22] in the case that $\xi \in \mathbb{C}^\times$ satisfies $|\xi| \neq 1$ using analytic properties of the action of the R-matrix of $U_\xi(\tilde{\mathfrak{g}})$. An R-matrix free approach was given in [12, §8] for the case that ξ is not a root of unity. For $\xi = 1$, the theorem was proved in [11]. We now give a proof that works for any $\xi \in \mathbb{C}'\backslash\{q\}$, thus completing the proof of Theorem 4.40.

For a brief review of the theory of blocks of an abelian category see [22, §1]. The proof of Theorem 4.40 is immediate from the following two propositions.

Proposition 4.41. If $V \in \widetilde{\mathcal{C}}_\xi$ is indecomposable, then $V \in \widetilde{\mathcal{C}}^\gamma_\xi$ for some $\gamma \in \widetilde{\Gamma}_\xi$. In other words, $\widetilde{\mathcal{C}}_\xi = \bigoplus_{\gamma \in \widetilde{\Gamma}_\xi} \widetilde{\mathcal{C}}^\gamma_\xi$.

The proof of Proposition 4.41 in the case $\xi = \zeta$ is analogous to that given in [11, 12] and we omit the details (see [12, §8.4]).

Proposition 4.42. For every $\gamma \in \widetilde{\Gamma}_\xi$, the category $\widetilde{\mathcal{C}}_\xi^\gamma$ is an indecomposable abelian category.

We now prove Proposition 4.42 for $\xi \neq q$. It suffices to show that, given $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^+$ such that $\gamma_\xi(\boldsymbol{\lambda}) = \gamma_\xi(\boldsymbol{\mu})$, there exists a sequence of indecomposable modules $W_1, \dots, W_m$ satisfying:

(a) $V_\xi(\boldsymbol{\lambda})$ is a constituent of W_1 and $V_\xi(\boldsymbol{\mu})$ is a constituent of W_m,
(b) W_j and W_{j+1} have at least one irreducible constituent in common.

By Theorem 4.38 and Proposition 4.37, it suffices to prove this in the case $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^+_{\xi, I_\bullet}$. Thus, let $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_m \in \mathcal{P}^+_\xi$ be a sequence as in Lemma 2.7. If $\xi = q$, it follows from Proposition 4.39 that the sequence $W_j = W_q(\boldsymbol{\omega}_j)$ satisfy the desired properties. If $\xi \in \mathbb{C}^\times$, set $\boldsymbol{\omega}'_1 = \boldsymbol{\lambda}$ and define $\boldsymbol{\omega}'_j \in \mathcal{P}^+_q, j > 1$, recursively by

$$\boldsymbol{\omega}'_{j+1} = \boldsymbol{\omega}'_j \left(\boldsymbol{\tau}_{q,k_j,a_j}\right)^{-\epsilon_j}. \tag{4.21}$$

Set $\boldsymbol{\mu}' = \boldsymbol{\omega}'_m, W'_j = W_q(\boldsymbol{\omega}'_j)$ and $W_j = \overline{W'_j}$. By Proposition 4.39, W'_j and W'_{j+1} have common irreducible constituents and, hence, so do W_j and W_{j+1} by Proposition 4.37. Since W_1 is a quotient of $W_\xi(\boldsymbol{\lambda})$ and W_m is a quotient of $W_\xi(\boldsymbol{\mu})$ we are done.

In the case $\xi = 1$, the proof of Proposition 4.42 we gave above is an alternative one to that given in [11].

4.7. **On the ℓ-character of fundamental modules.** We now verify that the main result of [12, §6] holds in the roots of unity as well. Throughout this subsection we assume $\xi \neq 1$. The next lemma is well-known and can be easily checked using the formulas of Lemma 4.14 (cf. [38, Proposition 9.2]).

Proposition 4.43. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda \in P_l^+$. Then $V_\zeta(\lambda) \cong W_\zeta(\lambda)$. □

Proposition 4.44. Let $\mathfrak{g} = \mathfrak{sl}_2, I = \{i\}, a \in \mathbb{C}(\xi)^\times, \lambda \in P^+$, and $r = \lambda(h_i)$. Then,

$$\mathrm{char}_\ell(W_\xi(a, \lambda)) = \boldsymbol{\omega}_{i,a,r} \sum_{k=0}^{r} \left(\prod_{j=1}^{k} \boldsymbol{\omega}_{i,a\xi^{r-2j},2}\right)^{-1} = \boldsymbol{\omega}_{i,a,r} \sum_{k=0}^{r} \left(\prod_{j=1}^{k} \boldsymbol{\alpha}_{i,a\xi^{r-2j+1}}\right)^{-1}.$$

Proof. For ξ not a root of unity this was proved in [26] (see also [12, Proposition 5.11]). The root of unity case then follows follows from Proposition 4.4 (see also [25]). □

Lemma 4.45. Let $V \in \widetilde{\mathcal{C}}_\xi$, $\boldsymbol{\mu} \in \mathcal{P}_\xi$, and suppose that there exist a nonzero $v \in V_{\boldsymbol{\mu}}$ and $j \in I$ such that $x^+_{j,s} v = 0$ for all $s \in \mathbb{Z}$. Then, $\boldsymbol{\mu}_j(u)$ is a polynomial of degree $\mathrm{wt}(\boldsymbol{\mu})(h_j)$ and $(x^-_{j,0})^{\mathrm{wt}(\boldsymbol{\mu})(h_j)} v \in V_{T_j \boldsymbol{\mu}} \backslash \{0\}$. Also, if the ξ_j factorization of μ_j is given by

$$\mu_j = \prod_{r=1}^{k} \boldsymbol{\omega}_{j,a_r,m_r},$$

then $\boldsymbol{\mu}(\boldsymbol{\alpha}_{j,a_r\xi^{m_r-1}})^{-1} \in \mathrm{wt}_\ell(V)$ for all $1 \le r \le k$. Furthermore, for all $s \in \mathbb{Z}$, we have

$$x^-_{j,s} v \in \sum_{r=1}^{k} \sum_{p=0}^{m_r-1} V_{\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_r\xi^{m_r-1-2p}}},$$

and

$$\dim(V_{\boldsymbol{\mu}\boldsymbol{\alpha}^{-1}_{j,a_r\xi^{m_r-1}}}) \ge \#\{1 \le s \le k : a_r = a_s\}.$$

Proof. The first two statements have already been proved. The remaining statements for ξ of infinite order were proved in [12, Proposition 6.5]. In the root of unity case the proof can be carried out exactly as in the infinite order case after observing the following. Due to Lemma 1.10 we can work with elements $\tilde{h}_{i,r}$ as it was done in [12, Proposition 6.5]. Moreover, observe as well that $\frac{[rc_{ij}]_{q_j}}{[r]_{q_i}} \in \mathbb{A}$ for all $i, j \in I$ and all nonzero integer r. Hence, we have the following identity in $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$

$$(q_j^r + q_j^{-r})\ [\tilde{h}_{i,r}, (x_{j,s}^-)^{(k)}] = \frac{[rc_{ij}]_{q_j}}{[r]_{q_i}}\ [\tilde{h}_{j,r}, (x_{j,s}^-)^{(k)}]$$

Finally, observe that $\zeta_j^r + \zeta_j^{-r} \neq 0$ for all $r \in \mathbb{Z}$ and set

$$\{rc_{ij}\}_\zeta = \epsilon_\zeta \left(\frac{[rc_{ij}]_{q_j}}{[r]_{q_i}} \right).$$

Hence, we get the following identity in $U_\zeta(\tilde{\mathfrak{g}})$

$$[\tilde{h}_{i,r}, (x_{j,s}^-)^{(k)}] = (\zeta_j^r + \zeta_j^{-r})^{-1} \{rc_{ij}\}_\zeta\ [\tilde{h}_{j,r}, (x_{j,s}^-)^{(k)}] \tag{4.22}$$

which replaces equation (6.19) of [12]. From here, all the steps of the proof can be performed exactly as in the case of ξ generic. $\square$

Given $\lambda \in P^+$, let $I(\lambda) = \{i \in I : \lambda(h_i) = 0\}$ and let $\mathcal{W}(\lambda)$ be the subgroup of $\mathcal{W}$ generated by $\{s_i : i \in I(\lambda)\}$. The proof of the following lemma can be found in [30].

Lemma 4.46. Let $\lambda \in P^+$. Then,

(a) $\mathcal{W}(\lambda) = \{w \in W : w\lambda = \lambda\}$.
(b) Each left coset of $\mathcal{W}(\lambda)$ in $\mathcal{W}$ contains a unique element of minimal length.
(c) Denote by $\mathcal{W}_\lambda$ the set of all left coset representatives of minimal length, suppose that $w \in \mathcal{W}_\lambda$, and that $w = s_j w'$ for some $w' \in \mathcal{W}$ with $\ell(w') = \ell(w) - 1$. Then, $w' \in \mathcal{W}_\lambda$. $\square$

The following theorem can now be proved exactly as in [12, Theorem 6.1].

Theorem 4.47. Suppose $\mathfrak{g}$ is of classical type. Let $i \in I$, $a \in \mathbb{C}(\xi)^\times$ and assume that $\boldsymbol{\lambda} \in \mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{i,a}))$ is such that $\mathrm{wt}(\boldsymbol{\lambda}) = \lambda \in P^+$. Then,

$$\dim(V_\xi(\boldsymbol{\omega}_{i,a})_{\boldsymbol{\lambda}}) = \dim(V_\xi(\boldsymbol{\omega}_{i,a})_{T_w \boldsymbol{\lambda}}) \qquad \text{and} \qquad T_w(\mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{i,a})_\lambda)) = \mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{i,a})_{w\lambda})$$

for all $w \in \mathcal{W}_\lambda$. Suppose further that $\boldsymbol{\lambda} \neq \boldsymbol{\omega}_{i,a}$. Then, there exist $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{i,a})), b, c \in \mathbb{C}(\xi)^\times$, and $j \in I$ such that $\boldsymbol{\mu}_j(u) = (1 - bu)(1 - cu)$, and

$$\boldsymbol{\lambda} = \boldsymbol{\mu}(\boldsymbol{\alpha}_{j,b})^{-1}. \tag{4.23}$$

Moreover, if $c \neq b\xi^{-2}$, then $\boldsymbol{\mu}\ (\boldsymbol{\alpha}_{j,c})^{-1} \in \mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{i,a}))$ and, if $c = b$, then $\dim(V_\xi(\boldsymbol{\omega}_{i,a})_{\boldsymbol{\lambda}}) \geq 2$. $\square$

Theorem 4.47 provides an algorithm for computing a lower bound for $\mathrm{char}_\ell(V_\xi(\boldsymbol{\omega}_{i,a}))$. For ξ of infinite order, it was used together with the knowledge of $\mathrm{char}(V_\xi(\boldsymbol{\omega}_{i,a}))$ (see [7]) in [13] to compute $\mathrm{char}_\ell(V_\xi(\boldsymbol{\omega}_{i,a}))$. Notice that it follows from Proposition 4.4 that $\epsilon_\xi(\mathrm{char}_\ell(V_q(\boldsymbol{\omega}_{i,a})))$ is an upper bound for $\mathrm{char}_\ell(V_\xi(\boldsymbol{\omega}_{i,a}))$. We finish the paper by giving an example explaining how to use these facts and Corollary 4.5 to identify values of ξ for which

$$V_\xi(\boldsymbol{\omega}_{i,a}) \cong \overline{V_q(\boldsymbol{\omega}_{i,a})}. \tag{4.24}$$

It follows from Proposition 4.4 that (4.24) holds iff

$$\mathrm{char}_\ell(V_\xi(\boldsymbol{\omega}_{i,a})) = \epsilon_\xi(\mathrm{char}_\ell(V_q(\boldsymbol{\omega}_{i,a}))). \tag{4.25}$$

Remark. Observe that, if ω_i is minuscule, then (4.24) holds for any value of ξ (including $\xi = 1$). In fact, given $a \in \mathbb{C}(\xi)$ and a minuscule ω_i, we have

$$\mathrm{char}_\ell(V_\xi(\boldsymbol{\omega}_{i,a})) = \sum_{w \in \mathcal{W}_{\omega_i}} T_w(\boldsymbol{\omega}_{i,a}).$$

On the other hand, if ω_i is not minuscule and $\xi = 1$, then (4.24) is always false because $V(\boldsymbol{\omega}_{i,a})$ is irreducible as $\mathfrak{g}$-module, while $\overline{V_q(\boldsymbol{\omega}_{i,a})}$ is not.

Example 4.48. Assume from now on that $\xi = \zeta, a \in \mathbb{C}^\times$, $\mathfrak{g}$ is of type D_n and $i = 2$. Let $\boldsymbol{\mu}_j \in \mathcal{P}_q$ be defined by

$$\boldsymbol{\mu}_j = \begin{cases} (\boldsymbol{\omega}_{j-1,aq^{j+1}})^{-1}\boldsymbol{\omega}_{j-1,aq^{2n-j-3}}\boldsymbol{\omega}_{j,aq^j}(\boldsymbol{\omega}_{j,aq^{2n-j-2}})^{-1}, & \text{if } 1 \le j \le n-2, \\ \boldsymbol{\omega}_{j,aq^{n-3}}(\boldsymbol{\omega}_{j,aq^{n+1}})^{-1}, & \text{if } j = n-1, n. \end{cases} \tag{4.26}$$

Thus, according to [12, (6.8)],

$$\mathrm{char}_\ell(V_q(\boldsymbol{\omega}_{2,a})) = \sum_{w \in \mathcal{W}_{\omega_2}} T_w(\boldsymbol{\omega}_{2,a}) + \sum_{j \ne n-2} \boldsymbol{\mu}_j + 2\boldsymbol{\mu}_{n-2}. \tag{4.27}$$

Since $\mathrm{wt}(T_w(\boldsymbol{\omega}_{2,a})) \notin P^+$, by Corollary 4.5, (4.24) holds provided $\overline{\boldsymbol{\mu}}_j \notin \mathcal{P}^+$ for all $j = 1, \dots, n$. If $1 < j < n-1$, $\overline{\boldsymbol{\mu}}_j \in \mathcal{P}^+$ iff l divides $2(n-j-2)$ and $2(n-j-1)$ which implies l divides 2. Similarly, $\overline{\boldsymbol{\mu}}_{m-1}, \overline{\boldsymbol{\mu}}_n \in \mathcal{P}^+$ iff l divides 4. Since we are assuming l is odd, we conclude $\overline{\boldsymbol{\mu}}_j \notin \mathcal{P}^+$ for all $j > 1$. Finally, $\overline{\boldsymbol{\mu}}_1 \in \mathcal{P}^+$ iff l divides $2(n-2)$. Hence, since l is odd, if l is not a divisor of $n-2$, (4.24) holds. We now compute the multiplicity of the space $V(\boldsymbol{\omega}_{2,a})_{\overline{\boldsymbol{\mu}}_j}$ assuming l does not divide $n-2$. Notice that, if $j < k$, then

$$\overline{\boldsymbol{\mu}}_j = \overline{\boldsymbol{\mu}}_k \quad \text{iff} \quad 1 \le j = k-1 < n-2 \quad \text{and} \quad l \quad \text{divides} \quad n-2-j. \tag{4.28}$$

Hence, $\overline{\boldsymbol{\mu}}_{n-3} \ne \overline{\boldsymbol{\mu}}_{n-2}$ for any value of l and

$$\dim(V_\zeta(\boldsymbol{\omega}_{2,a})_{\overline{\boldsymbol{\mu}}_j}) = \begin{cases} 2, & \text{if } 1 \le j \le n-2 \text{ and } l \text{ divides } n-2-j, \\ 1, & \text{otherwise.} \end{cases} \tag{4.29}$$

Let us now discuss the case when l divides $n-2$. As pointed out to us by Nakajima, it follows from the algorithm given in [42] that

$$\mathrm{char}_\ell(V_\zeta(\boldsymbol{\omega}_{2,a})) = \sum_{w \in \mathcal{W}_{\omega_2}} T_w(\boldsymbol{\omega}_{2,a}) + \sum_{j \ne 1, n-2} \overline{\boldsymbol{\mu}}_j + 2\overline{\boldsymbol{\mu}}_{n-2}. \tag{4.30}$$

Moreover, $\dim(V_\zeta(\boldsymbol{\omega}_{2,a})_{\overline{\boldsymbol{\mu}}_j})$ for $j \ne 1$ is given by (4.29). We now give an alternate proof of (4.30) using Theorem 4.47 and the theory of specialization of modules. Let ξ be any element of $\mathbb{C}'$ again. As in [12, §6.5], set

$$w_j = s_{j-1} \cdots s_1 s_{j+1} \cdots s_{n-2} s_n s_{n-1} s_{n-2} \cdots s_1.$$

Observe that $\mathcal{W}_{\omega_2} = \{w_j : j = 1, \dots, n\}$ and $w_j \omega_2 = \alpha_j$. One then computes (cf. [12, (6.10)])

$$T_{w_j}(\boldsymbol{\omega}_{2,a}) = \begin{cases} (\boldsymbol{\omega}_{j-1,a\xi^{j+1}})^{-1}\boldsymbol{\omega}_{j,a\xi^j}\boldsymbol{\omega}_{j,a\xi^{2n-4-j}}(\boldsymbol{\omega}_{j+1,a\xi^{2n-3-j}})^{-1}, & \text{if } j \le n-2, \\ (\boldsymbol{\omega}_{n-2,a\xi^n})^{-1}\boldsymbol{\omega}_{j,a\xi^{n-1}}\boldsymbol{\omega}_{j,a\xi^{n-3}}, & \text{if } j = n-1, n. \end{cases} \tag{4.31}$$

Notice

$$\overline{\boldsymbol{\mu}}_j = \begin{cases} T_{w_j}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{j,a\xi^{2n-4-j}})^{-1}, & \text{if } 1 \le j \le n-2, \\ T_{w_{j-1}}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{j-1,a\xi^{j-1}})^{-1}, & \text{if } 2 \le j \le n-2, \\ T_{w_j}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{i,a\xi^{n-3}})^{-1}, & \text{if } j = n-2, i = n-1, n, \\ T_{w_j}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{j,a\xi^{n-1}})^{-1}, & \text{if } j = n-1, n. \end{cases} \tag{4.32}$$

It immediately follows from Theorem 4.47 that $\overline{\boldsymbol{\mu}}_j \in \mathrm{wt}_\ell(V_\xi(\boldsymbol{\omega}_{2,a}))$ for $j \geq n-2$ and, moreover, that $\dim(V_\xi(\boldsymbol{\omega}_{2,a})_{\overline{\boldsymbol{\mu}}_{n-2}}) \geq 2$. Let $1 < j \leq n-2$ and observe that

$$\frac{\zeta^{2n-4-j}}{\zeta^j} = \zeta^{-2} \quad \Rightarrow \quad \frac{\zeta^{j-1}}{\zeta^{2n-3-j}} \neq \zeta^{-2}. \tag{4.33}$$

Together with Theorem 4.47, this implies that, if $1 < j \leq n-2$, either $T_{w_j}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{j,a\xi^{2n-4-j}})^{-1} \in \mathrm{wt}_\ell(V_\zeta(\boldsymbol{\omega}_{2,a}))$ or $T_{w_{j-1}}(\boldsymbol{\omega}_{2,a})(\boldsymbol{\alpha}_{j-1,a\xi^{j-1}})^{-1} \in \mathrm{wt}_\ell(V_\zeta(\boldsymbol{\omega}_{2,a}))$. Hence, $\overline{\boldsymbol{\mu}}_j \in \mathrm{wt}_\ell(V_\zeta(\boldsymbol{\omega}_{2,a}))$ for all $j > 1$. It remains to show that $\overline{\boldsymbol{\mu}}_1 \notin \mathrm{wt}_\ell(V_\zeta(\boldsymbol{\omega}_{2,a}))$ if l divides $n-2$.

To do this, let $V = V_q(\boldsymbol{\omega}_{2,a})$, v be a highest-ℓ-weight vector of V, $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$, and v_{w_j} be as in (4.7). In particular, v_{w_j} is an $\mathbb{A}$-basis element of L and, hence, its image $\overline{v_{w_j}} \in \overline{V}$ is nonzero. Let also $W_j = U_q(\tilde{\mathfrak{g}}_j)v_{w_j}$ and $\overline{W_j} = U_\zeta(\tilde{\mathfrak{g}}_j)\overline{v_{w_j}}$. Since v_{w_j} is a highest ℓ-weight vector for $U_q(\tilde{\mathfrak{g}}_j)$, W_j is a quotient of the Weyl module for this subalgebra with highest ℓ-weight $(T_{w_j}(\boldsymbol{\omega}_{2,a}))_j$ (and similarly for $\overline{W_j}$). It follows from the above computations that $V_{\boldsymbol{\mu}_j} \subseteq W_j$ and that $V_0 = \sum_{j=1}^n (W_j)_0$. We claim that W_1 must be a Weyl module. In fact, it is easy to see that $V_{\boldsymbol{\mu}_1} \cap W_j \neq \{0\}$ iff $j = 1$. If W_1 were not a Weyl module, then it would be irreducible and so would be $\overline{W_1}$. This would imply that $(\overline{\boldsymbol{\mu}}_1)_1$ (which is the the constant polynomial) would be an ℓ-weight of the irreducible $U_\zeta(\tilde{\mathfrak{sl}}_2)$-module with highest ℓ-weight $(1 - a\zeta u)(1 - a\zeta^{-1}u)$. This contradicts Proposition 4.44.

Since v_{w_1} is an $\mathbb{A}$-basis element of L, it follows that $\overline{W_1}$ is also a Weyl module (alternatively, if $\overline{W_1}$ were not a Weyl module, $\overline{V}$ would not be highest-ℓ-weight since $\overline{V}_{\overline{\boldsymbol{\mu}}_1} \cap \overline{W_j} = \{0\}$ if $j > 1$ and we would have a contradiction again). Therefore, there exists $v' \in (\overline{W_1})_0$ which generates a trivial submodule of $\overline{W_1}$. It is easy to see that v' also generates a trivial submodule of $\overline{V}$. This shows that $\overline{\boldsymbol{\mu}}_1$ is not an ℓ-weight of the irreducible quotient of $\overline{V}$ which is isomorphic to $V_\zeta(\boldsymbol{\omega}_{2,a})$.

Remark. The assumption that l is odd is not really essential above and one can easily deduce similar results for roots of unity of even order as well.

We believe that the above line of reasoning could be used alongside the results of [13] to obtain expressions for the ℓ-characters of the fundamental representations of $U_\zeta(\tilde{\mathfrak{g}})$ in terms of the braid group action for $\mathfrak{g}$ of classical type. In order to keep the length of present text within reasonable limits, we postpone further discussion in this direction to a forthcoming publication.

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